MARKET CONSISTENT VALUATION UNDER THE SOLVENCY II DIRECTIVE

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INTRODUCTION

This thesis is divided into two parts. The first part involves the new solvency directive for the insurance industry in the European Union, Solvency II, which will be implemented in 2012. The second part involves valuation strategies under the new directive.

First, I will investigate the concept of solvency and introduce the current solvency directive, the Solvency I directive. Then I investigate the Solvency II directive especially for life insurance undertakings within the European Union, which Norway have agreed to follow. I describe the technical specifications of the Solvency II directive, including its definitions and formulas, and discuss modifications which have to be implemented for Norwegian life insurance undertakings. The goal of the new directive is to achieve stability for the financial system. This shall be done by having enough buffer capital on hand to cover expected and unexpected losses. The insurance industry has had its deal of problems throughout the financial crisis. Factors include poor growth in the stock market, and low interest rates which in turn have eaten up a large part of their buffer capital. We therefore see a need for stricter requirements and regulations of the industry.

The Solvency II directive requires that the undertakings valuate both the asset side, and liability side in a market consistent way. The second part of the thesis starts out by explaining the cash flows involved in life insurance. I continue by discussing different risk neutral discount factors which are mentioned in the Solvency II directive. I then introduce a method of achieving a market consistent valuation of both the asset side, and the liability side. This method links the insurance cash flow to financial instruments. Then by finding the market values of the financial instruments one can find the value of the insurance portfolio. I look at a numerical example, to show this valuation method in practice. In the end I introduce two asset liability management strategies which can be used, and still remain solvent.

2

SOLVENCY

Solvency is an old concept. According to the Miriam-Webster Dictionary solvency originates from ca. 1727 as

> the quality or state of being solvent.

A company is regarded solvent if it is

> able to pay all its legal debts [by the mature date].

If the solvency of a company is good, it has financial strength. If on the other hand a company is insolvent, the company can no longer operate and files for bankruptcy. We will only consider insurance companies. An insurance company's legal debts are its expected insurance liabilities. Therefore a company needs to hold assets which covers these liabilities. If the company invests in financial instruments it will have to hold assets to cover the risk carried from the financial investments as well. The definition of financial risk is the probability of unfavorable events, e.g. the probability that the actual return becomes less what is expected return. The capital buffer needed for a company to become solvent is called the solvency requirement. The big questions regarding solvency of a company are:

- How large should the solvency requirement be?
- What time horizon should one consider?
- What kinds of assets can be used to cover the solvency requirement?

For regulatory system purposes these questions are answered both in the Solvency I and again in the Solvency II directive. The answers in the two directives are designed to minimize the likelihood of failure, while minimizing the costs to policyholders in the event of failure. They are not a "zero-failure" regime. It is not realistic to build such a system where we can guarantee that no insurance undertaking will ever fail. The insurance undertakings would have to hold unlimited capital in order to cover extremely unlikely, yet devastating event. The protection comes at a cost, the higher the level of guarantee, the higher the cost to he policyholders and the economy as a whole. Therefore a balance has to be struck to offer affordable and safe insurance products.

2.1 Purpose of solvency

The purpose of solvency is to enable the investor to tell whether a company can pay its debts or not. As the insurance companies are growing into large investors, their financial strength is becoming very important for the soundness of the entire financial market. The International Association of Insurance Supervisors (IAIS) is an organization which works to promote financial stability and the development of well-regulated insurance markets. They were established in 1994, and represents insurance regulators and supervisors in nearly 140 countries. The IAIS defines solvency as:

> the ability of an insurer to meet its liabilities under all contracts at any time. Due to the very nature of insurance business, it is impossible to guarantee solvency with certainty. In order to come to a practicable definition, it is necessary to make clear under which circumstances the appropriateness of the assets to cover claims is to be considered

The capital buffer must consist of free assets which can be realized within reasonable time.

2.2 Solvency I

The Solvency I directive is the current solvency directive used by the insurance undertakings of the EU member states. Solvency I is from 2002 and consists of 74 articles. The solvency requirement in this directive is formed to ensure that an insurers' capital can act as a buffer against adverse business fluctuations.

The Solvency I consist of two capital requirements for life insurance. The Available Solvency Margin (ASM) shall consist of the assets of the insurance undertaking free of any foreseeable liabilities. It does not specify how to valuate the assets. This is left for each member state to decide. The Required Solvency Margin (RSM) is in the robust form

$$\begin{aligned} \text{RSM} = {} & 4\% \text{ of market risk} \\ & + 0.3\% \text{ of technical risk} \end{aligned} \tag{2.1}$$

Market risk is the risk carried from the investment portfolio in the financial market, e.g. the risk of changes in stock prices or interest rates. Technical risk is also called insurance risk and arise from using the wrong claim rates e.g. the mortality rates may be incorrect. The solvency rules in the Solvency I directive are simple, and rule-based. This makes the rules easy to understand, but do not take into account what kinds of businesses that is written, and neglect differences between the asset and liability profile.

Already during the forming of the Solvency I directive it was concluded that further attention had to be placed on the developments in the financial service industry. The working group that proposed Solvency I suggested that

> it may be desirable to undertake a wider review of the EU solvency requirement system and to consider whether more explicit recognition of the different risks is required

Their further work developed into a new solvency directive, Solvency II. Switzerland chose a different approach, and adopted a risk-based capital system for the solvency requirement on their own initiative. Current Norwegian requirements for solvency can be found under www.lovdata.no[1], and follow the regulations of the current EU directive.

[1]Forskrift om beregning av solvensmarginkrav og solvensmarginkapital for norske livsforsikringsselskaper from 1995, §3 and §4.

3

SOLVENCY II

Solvency II is a new set of regulatory requirements for insurance undertakings that operate in the EU. The motivation for EUs insurance legislation is to ease the development of a Single Market in insurance services in Europe, while at the same time securing an adequate level of protection. The Solvency II directive has been in development for quite some time. There is still a lot of work to be done before the new directive is implemented. But the framework of the directive was approved at the EU parliament April 22 2009 and the final text adopted by the Council November 10 2009. When I started writing this thesis the final Solvency II directive had not yet been stated. Nevertheless I have described and commented on the framework of the directive and assumed that the final text would not differ a lot from the framework. Fortunately my predictions were correct.

The Solvency II directive consists of several old directives merged to one and includes more than 300 articles. As the commission states,

> the new directive is a recast, not a complete rewriting.

Even though about half of the articles of the new directive remains unchanged, the most central articles are altered. These central articles include capital requirements, solvency requirements and asset management. There is also new requirements concerning governance and supervision, internal assessment of risk and solvency, capital add-ons and public disclosure of information.

3.1 History

In the past years there have been a number of failures and “near misses” in the European finance industry. A group of supervisors investigated these incidents between 1996 and 2001, and described cause-effect mappings and diagnostics for a total of 21 cases. A report was made in 2002 called the Sharma report. The conclusions were that a solvency directive should include requirements for governance and risk management within the companies, and the undertakings should build tools to monitor and mitigate risks at all levels. In most of the studied cases, there was a chain of multiple underlying causes for failure. So, the report stated that

> Capital is only the second strategy of defense in a company, the first is good risk management

The new Solvency II directive is, as we shall see, risk adjusted and based on a market consistent valuation of both assets and liabilities.

Even before the financial crisis burst out, some companies, including insurance undertakings, already concluded that the current requirements in the existing solvency directive were insufficient. Some even implemented their own reforms. For example, in Switzerland the Swiss implemented their own Swiss Solvency Test and demanded that all Swiss insurance companies comply with it. The Swiss Solvency test is similar to EU's Solvency II directive, and is so thorough that Switzerland is not going to implement EUs new solvency directive. The implementation of own reforms throughout Europe has lead to a patchwork of regulatory requirements. This hinders EU's intent of developing a Single Market.

The new Solvency II rules will replace all the old requirements, and apply to all insurance undertakings by the end of October 2012. Norway has agreed to follow the Solvency II directive.

3.2 Goals

The goal of the Solvency II directive is to protect policyholders, and the stability of the financial system as a whole. It encourages undertakings to measure and manage their own risk. It defines the quality of assets, and the minimum amounts of financial resources insurers must have to cover liabilities and risks. Actuaries now must look at the "total balance sheet", instead of just concentrating on technical risk. In order to do so they should valuate both assets and liabilities by a common market value. In addition to managing their own risk, insurers are required to disclose key information to supervisors and market participants. This leads to an early warning system, where supervisors will have more time to make demands to improve and restore a companys financial health.

3.3 Main points

The Solvency II directive is inspired by the Basel II accord from the banking industry, which was introduced in 2006.

3.3.1 The Pillars

As the Basel II accord, the Solvency II directive is formulated in three main pillars. Pillar 1 focuses on the quantitative capital requirements. Pillar 2 focuses on the qualitative requirements such as a supervisory review process. Pillar 3 focuses on disclosure requirements and greater market discipline.

3.3.1.1 Pillar 1

The first pillar specifies two financial requirements needed to obtain solvency, the Solvency Capital Requirement (SCR) and the Minimum Capital Requirement (MCR).

The SCR reflects the capital an insurance undertaking must have available to cover all its risks. The SCR can be determined either by the European Standard Formula which I will elaborate on later, or by an internal model approved by the local financial supervisory authorities (FSA). Companies are encouraged to develop internal models, to improve their own risk management and governance depending on how they assess the risks. The internal model might decrease the capital requirements. They might also cause an even higher capital requirement than that of the standard formula.

The MCR is an absolute lower capital requirement. It corresponds to the RSM of the Solvency I directive. This requirement constitutes a basic trigger mechanism for ultimate supervisory action, which leads to closure to new business or withdrawal of authorization.

One of the consequences of the pillar 1 is that insurers now are allowed to invest in any asset they wish, provided that they can demonstrate that they understand the underlying risk involved, and have capital enough to carry the risk.

3.3.1.2 Pillar 2

The second pillar specifies requirements for good internal risk management. The insurer should have an internal assessment process for their assets, and show that they understand the financial instruments their assets are invested in. The insurer must also have strategies to manage the risks they are subject to. The risks involved will now include market risk, credit risk and operational risk.

- *Market risk* is the fall in the value of insurers' investments on the financial market.
- *Credit risk* is also called the default risk, and is the risk that third parties are unable to repay their debts to the insurer.
- *Operational risk* is the risk of losses resulting from inadequate or failed internal processes caused by people, systems and from external events.

These are all risks that are not thoroughly covered by the current EU regime. The insurance underwriting risk, also called technical risk, is the risk that the premiums will not cover the future incurred losses such that the current reserves are insufficient. In life insurance longevity is an example of an underwriting risk the insurers must be aware of, i.e. that the population lives longer than expected.

3.3.1.3 Pillar 3

The Solvency II directive requires that the participating countries appoint a financial supervisory authority that can evaluate the internal governance system and the capacity of the management of the country's insurance undertakings. In Norway the financial supervisory authority will be represented by Finanstilsynet. Pillar 3 obligates firms to disclose key information publicly. This enables supervisors to evaluate the information and require adjustments where necessary so that policyholders can feel reassured. This disclosure also provides every competitor with key information about the other companies in the industry, a fact that was probably not the intent of the directive. The main reason of reporting to supervisors is that they can identify insurers who might be heading for difficulties. The supervisors should then have the proper power and means to take preventive and corrective measures to ensure that the undertakings comply with the requirements of the directive.

3.4 Valuation

The Solvency II risk-based philosophy for determining solvency capital requirements aims to take account of all potential risks faced by the insurance undertakings. This includes insurance, market, credit and operational risk. In line with the directive proposal, this assessment should be made using an economic, market-consistent valuation of all assets and liabilities.

This implies that whenever it is possible, the undertaking shall use the fair value, called the mark-to-market method, for valuation. This involves decomposing the liability cash flow into units and linking them to financial instruments. In that way one can use the current arbitrage free market price for the instrument to valuate the liability cash flow. I will explore this valuation method further in chapter 7.

If it is not possible to use a fair-value method for valuation, a mark-to-model method should be used. This can for example occur in incomplete markets where there simply does not exist replicating strategies for liabilities. To replicate the liabilities means to construct a portfolio with the same value using instruments from the financial market. The mark-to-model method is defined as any kind of valuation which has to be benchmarked, extrapolated or otherwise calculated from a market model rather than market prices.

The SCR shall be calculated so that it is able to cover all losses with a confidence level of 99.5%. To calculate the SCR it is therefore necessary to find the risk for losses both from the insurance side and from the financial investments involved in the company's liabilities. A detailed explanation of the calculation of the SCR is described in section 3.6.2.

3.5 Technical provisions

Technical provisions are defined as the capital needed to meet the obligations the company has towards the policyholders of insurance contracts. Article 75 in the directive states that

> the value of technical provisions shall correspond to the current amount insurance and reinsurance undertakings would have to pay if they were to transfer their insurance and reinsurance obligations immediately to another insurance or reinsurance undertaking

The technical provisions are further divided into two elements, the discounted best estimate cash flow, and a risk margin.

3.5.1 Best estimate

The discounted best estimate cash flow is the discounted probability-weighted average of the future cash flow of claims and expenses, using the relevant risk-free interest rate term structure. Under the Solvency II directive the risk-free interest rate is to be computed by the swap rate. If there does not exist a swap rate market, government bonds are to be used. The calculation of the best estimate cash flow should be based on current information and realistic assumptions, and be performed using adequate actuarial and statistical methods. The best estimate cash flow projection should take into account all the cash flows required to settle the obligations over a lifetime. I will elaborate on this calculation in chapter 5.

3.5.2 Risk margin

The risk margin is the extra capital which ensures that the technical provisions are equivalent to the capital another insurance undertaking would require in order to take over all of the undertaking's obligations. In the Solvency II directive the risk margin capital is equal to the cost of providing the amount of eligible own funds needed to support the insurance obligations over their lifetime.

3.5.3 Valuation of the technical provisions

According to Groupe Consultatifs Solvency II Pillar I working group, the technical provisions would ideally be established as the best estimate discounted reserves plus a market value margin based on the market cost of hedging, see chapter 9. The Groupe Consultatif's purpose is to bring together the actuarial associations in the EU to represent the actuarial profession in discussion with the EU's institutions on EU legislation which has an impact on the profession.

The best estimate and the risk margin are to be valued separately. The exception is when the future cash flow associated with insurance obligations can be replicated using financial instruments for which market values can be found. If so the value of the technical provisions should be determined by the market value of those instruments comprising the technical provisions.

3.6 SCR and MCR

The Solvency II directive is principle-based which means no formulas are included directly in the directive. The formulas are only supplemented in consultation papers. The consultation papers provide a standard formula for the overall SCR for the undertaking. In addition they define capital charge, or SCR, for operational risk as well as a correlation table between the capital charges for the five main risk categories namely market risk, life underwriting risk, counterparty default risk, health underwriting risk, and non-life underwriting risk. The capital charge is the amount of capital they are required to have to support each risk module. The consultation papers also provide definitions, formulas and correlation tables for sub-modules within these five main risk categories.

But before we look at the calculation of the SCR and MCR let us look at the relationship between the capital requirements from the Solvency I directive and the Solvency II directive.

3.6.1 Relation between the capital requirements in Solvency I and Solvency II

The undertaking must have assets to cover the liabilities. Under the Solvency I directive, the capital requirement an undertaking needed to have in order to run the business was the Required Solvency Margin. This requirement has been divided into two different capital requirements in the Solvency II directive. In the new directive, the lower, and absolute minimum requirement is the Minimum Capital Requirement. It is calculated quite differently from the RSM in the Solvency I directive, see 2.1 and 3.2. The SCR is another capital requirement for the Solvency II directive established to help the supervisory authorities regulate the industry. This requirement is set higher than the RSM of the Solvency I directive. An illustration of the different capital requirements belonging to the two solvency directives is given in figure 3.1.

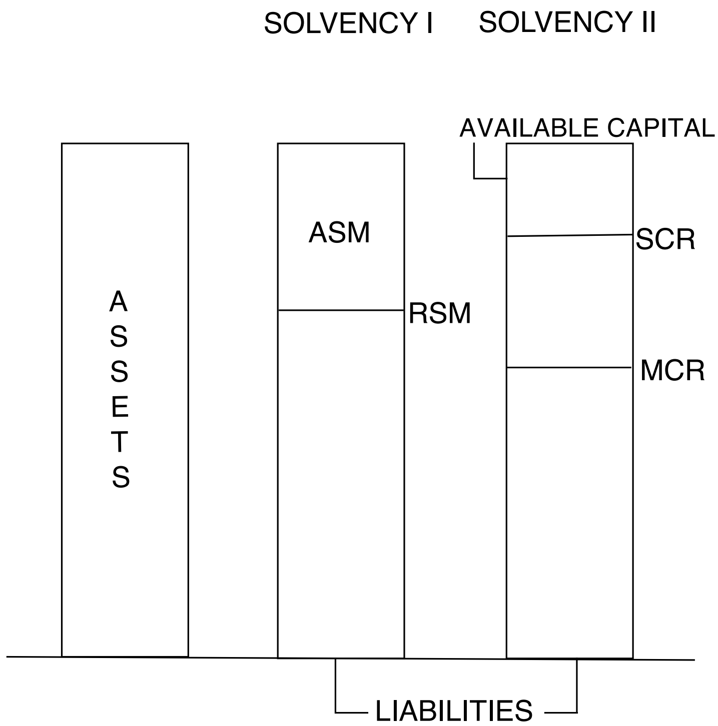

Figure 3.1: Relationship between the capital requirements

3.6.2 Calculation of the SCR and MCR

The SCR is calculated either with the standard formula given by the directive, or by an internal model. Both formulas are to be based on a Value at Risk measure (VaR) of the insurance liabilities calibrated at a 99.5% confidence level over a one year period. A VaR-measure is a commonly used risk measure and is used to measure the risk of loss on a specific portfolio of assets or liabilities. It measures the worst expected mark-to-market loss under normal conditions over a specific time period at a given confidence level. A 99.5% VaR over one year reflects the worst loss one would expect to occur in a single year, with the expectation that such a loss only happens one year in 200.

The MCR is not fully risk-based. It is calculated either by the standard formula with a VaR calibrated to a 85% confidence level over a period of one year, or by a tunnel between 25-50% of the SCR.

$$MCR_{def} = \min(\max(MCR_{85\%VaR}, 0.2 \cdot SCR), 0.5 \cdot SCR) \tag{3.1}$$

The confidence level approach is preferred by the industry for its simplicity, and provides an adequate ladder of intervention under the QIS 3 exercise. However, since it is not independent to the SCR it is considered that it does not provide a suitable 'safety net". The tunnel approach, on the other hand, has been criticized for not being sufficiently risk sensitive, it appears to be an evolution of the Solvency I approach. There is no consideration of the firm's risk posed by the assets it holds.

The MCR absolute minimum, MCR_{abs}, is 2.2M EUR for non-life insurance, and 3.2M EUR for life and reinsurance

$$MCR = \max(MCR_{def}, MCR_{abs}). \tag{3.2}$$

The MCR is to be computed quarterly, but the SCR annually. But since the MCR is dependent on the SCR, in reality both will have to be computed quarterly. In addition the capital requirements shall be computed again when the undertaking experiences changes in their risk profile.

3.6.3 Expected shortfall vs value at risk

Throughout the Solvency II directive, value at risk has been the chosen risk measure. The problem with the VaR measure is that it is not convex, i.e. it does not take into account the diversification effects of financial positions. An alternative risk measure which would be better to use is the expected shortfall. It can for example be used in internal models. In contrast to the VaR measure the expected shortfall measure takes convex risk measure diversification effects into account. The expected shortfall is more sensitive to the shape of the loss distribution in the tail of the distribution. Where the value at risk measure asks 'how bad can things get", the expected shortfall risk measure asks "if things go bad, how much can we expect to loose". The expected shortfall is the expected loss over one period conditioned on the fact that we are in the (100-X)% left tail of the distribution. The value at risk is used as the risk measure in the Solvency II directive since it is easy to understand and to implement. The expected shortfall is however not much more complicated, and is an even more conservative risk measure.

3.7 Standard Formula

The European Standard Formula is a basic calculation method that insurers can use to determine their overall SCR. The structure of the formula is set but the final calibration of the parameters are still in the testing phase. The aim of the standard formula is to differentiate and quantify each risk, i.e. market risk, credit risk, underwriting risk and operational risk. The parameters will be available at least 12 months before the insurers need to start applying the new rules. I will comment on how the standard formula is as described in the QIS 4 Technical Specifications. QIS 4 is the latest of the quantitative impact studies which have been held. The studies are used to calibrate the parameters of the Solvency II directive, see section 4.1.

The overall SCR, or capital charge, for an undertaking is defined as

$$\mathrm{SCR} = \mathrm{SCR}_{\mathrm{OP}} + \mathrm{BSCR} - \mathrm{Adj} \tag{3.3}$$

where

- SCR_{OP} = The capital charge for operational risk, which is the risk of loss arising from failed internal processes, people or systems. In other words, risk arising from human errors. Operational risk also include legal risk. It tries to address all risks which are not covered in the other risk modules, such as financial risk.
- BSCR = The basic SCR before adjustments, combining capital charges for the five major risk categories
- Adj = The sum of the adjustment for risk absorbing effect of future profit sharing and adjustments for risk absorbing effect of deferred taxes.

The capital charge of operational risk is calculated by

$$\mathrm{SCR}_{\mathrm{OP}} = \min(0.3 \cdot \mathrm{BSCR}, \mathrm{OP}_{\text{not ul}}) + 0.25 \cdot \mathrm{Exp}_{\mathrm{ul}} \tag{3.4}$$

where $\mathrm{OP}_{\text{not ul}}$ refers to the basic operational risk charge for all business other than unit-linked business and $\mathrm{Exp}_{\mathrm{ul}}$ is the amount of annual expenses incurred in respect of unit linked business. A unit linked insurance plan is a financial product where the policy value at any time varies according to the value of the underlying assets at the time. In other words, it offers both a life insurance as well as an investment such as a mutual fund. In unit linked businesses, the policyholder carries the financial risk of self chosen investments. Equation (3.4) restricts the $\mathrm{SCR}_{\mathrm{OP}}$ to a percentage of the other capital requirements included in the BSCR.

The BSCR is given by

$$\mathrm{BSCR} = \sqrt{\sum_{r\times c} \mathrm{CorrSCR}_{r,c} \cdot \mathrm{SCR}_r \cdot \mathrm{SCR}_c} \tag{3.5}$$

with $\mathrm{CorrSCR}_{r,c}$ corresponding to the cells of the correlation table 3.1. The SCR_r and SCR_c corresponds to the the capital charges for the individual risks which I will discuss below. The correlation parameters are subject to change before Solvency II implementation, but as of now they are as follows[1]

[1] The correlation matrix is symmetrical

Corr SCR	SCR_{market}	SCR_{default}	SCR_{life}	SCR_{health}	$SCR_{\text{non life}}$
SCR_{market}	1				
SCR_{default}	0.25	1			
SCR_{life}	0.25	0.25	1		
SCR_{health}	0.25	0.25	0.25	1	
$SCR_{\text{non life}}$	0.25	0.5	0	0	1

Table 3.1: Correlation table for overall SCR

Not all of the individual risks are relevant to life insurance, but the $\text{SCR}_{\text{market}}$, $\text{SCR}_{\text{default}}$ and the SCR_{life} are. I will describe these individual risks from the overall SCR. The parameters are determined to calibrate the VaR of the overall SCR to a confidence level of 99.5%.

3.7.1 Market risk

The market risk is the risk that arises from the volatility of the market prices of financial instruments. Examples of market risks are interest rate risk, equity risk, property risk, spread risk, risk concentration and currency risk. All the sub-modules of the market risk are determined so that the $\text{SCR}_{\text{market}}$ becomes the result of a stress test with all the worst case scenarios happening at the same time. The worst case scenarios corresponds to a defined shock to the net asset value minus all liabilities, NAV. That is to say the immediate expected effect on assets and liabilities given the event of a shock. An example of market risk arising from equity risk is

$$\text{Mkt}_{\text{equity}} = \Delta NAV|\text{equity shock} \tag{3.6}$$

Interest rate risk exists for all assets and liabilities for which the net asset value is sensitive to changes in interest rates or interest rate volatility. These assets include fixed-income investments, insurance liabilities and interest rate derivatives. The QIS 4 technical specification states that the undertakings should use the zero coupon swap curve as interest term structure. A swap curve is constructed from interest rates at which a fixed interest rate is swapped against the 6-month EURIBOR[2] which is a floating rate. It further states that the capital charge of the interest rate risk is determined by the ΔNAV given an upwards or downwards stress factor, s^{up} or s^{down}, to the current interest rates. The altered term structures are given by multiplying the current interest rate curve by $(1+s)^{\text{up}}$ or $(1+s)^{\text{down}}$. The stress parameters are specified for individual maturities. The capital charge for interest rate risk is derived from the shock that gives rise to the highest capital charge.

Equity risk arises from the volatility of market prices for equities. Exposure to equity risk refers to all assets and liabilities whose value is sensitive to changes in equity prices.

[2]European Interbank Offer Rate

The equity risk module uses indices as risk proxies, meaning that the volatility and correlation information is derived from these indices. The capital charge of equity risk is defined as ΔNAV given a stress factor on all of the indices. The stress factor varies between a 32% fall of equity indices listed in EEA[3] and OECD[4] countries, and a 40% fall for all other equity indices. Due to the financial crisis however we observed a fall in the indices in the first group of more than 40% in 2008.

The **property risk** is the risk that arises from volatility of market prices in the real estate market. A worst case scenario is a fall in property prices compared to the real estate benchmark. The capital charge for property risk is therefore the ΔNAV given a 20% decrease of the real estate benchmark.

The **currency risk** is the risk that the local currency will rise or fall respectively in value against all other currencies. The capital charge of the currency risk is ΔNAV given an upward or downward shock factor of 20%. The capital charge for currency risk is the shock that gives rise to the highest capital charge.

Spread risk is the risk of changes in the market value of for example bonds caused by changes in credit spreads. It reflects the change in the market value due to a move of the yield curve relative to the risk-free interest rate. Unit linked contracts, where the policyholder bears the investment risk, should be excluded from this module. The capital charge of the spread risk is given by the formula

$$\sum_i \mathrm{MV}_i \cdot m(\mathrm{dur}_i) \cdot F(\mathrm{rating}_i) \cdot \Delta\mathrm{Liab} \tag{3.7}$$

This formula include

- MV_i which is the credit risk exposure determined by reference to market values.
- $m(\mathrm{dur}_i)$ which is a function of the duration depending on rating class AAA-Unrated.
- $F(\mathrm{rating}_i)$ which is a function of the rating class itself, and gives the credit risk exposure calibrated to deliver a shock consistent with VaR of 99.5%.
- ΔLiab which is the overall impact on the liability side when the policyholders bear the investment risk.

Concentration risk arises from the additional volatility that exists in concentrated asset portfolios and the risk of partial or total permanent losses of value due to the default of an issuer. These parameters and functions are all included in the consulting papers.

Now that all the sub-module risks of the market risk are explained we can define the

[3]European Economic Area
[4]Organization of Economic Co-operation and Development

capital charge for market risk which is given by

$$\mathrm{SCR}_{\mathrm{market}} = \sqrt{\sum_{r\times c} \mathrm{CorrMkt}_{r,c} \cdot \mathrm{Mkt}_r \cdot \mathrm{Mkr}_c} \tag{3.8}$$

where the correlation table for market risk is as in table 3.2, where Mkt_r and Mkt_c corresponds to the the capital charges for the individual market risk modules discussed above

CorrMkt	$\mathrm{Mkt}_{\mathrm{int}}$	$\mathrm{Mkt}_{\mathrm{equity}}$	$\mathrm{Mkt}_{\mathrm{prop}}$	$\mathrm{Mkt}_{\mathrm{spread}}$	$\mathrm{Mkt}_{\mathrm{concentration}}$	$\mathrm{Mkt}_{\mathrm{currency}}$
$\mathrm{Mkt}_{\mathrm{int}}$	1					
$\mathrm{Mkt}_{\mathrm{equity}}$	0	1				
$\mathrm{Mkt}_{\mathrm{prop}}$	0.5	0.75	1			
$\mathrm{Mkt}_{\mathrm{spread}}$	0.25	0.25	0.25	1		
$\mathrm{Mkt}_{\mathrm{concentration}}$	0	0	0	0	1	
$\mathrm{Mkt}_{\mathrm{currency}}$	0.25	0.25	0.25	0.25	0	1

Table 3.2: Correlation table for market risk

3.7.2 Counterparty default risk

Counterparty default risk is the risk of possible losses due to an unexpected default or deterioration in the credit standing of counterparties in relation to risk mitigating[5] contracts. Default can occur in reinsurance arrangements and derivatives and the estimated losses are to be computed separately. The estimated loss given default of reinsurance contracts is given by

$$\mathrm{LGD} = 0.5 \cdot \max(\mathrm{Recoverables} + \mathrm{SCR}^{\mathrm{Gross}}_{\mathrm{U/W}} - \mathrm{SCR}^{\mathrm{Net}}_{\mathrm{U/W}} - \mathrm{Collateral}, 0) \tag{3.9}$$

where

- Recoverables is the expected amount of recoverable capital from the reinsurance contract
- $\mathrm{SCR}^{\mathrm{Gross}}_{\mathrm{U/W}}$ is the SCR for underwriting risk calculated by the standard formula, see section 3.7.3
- $\mathrm{SCR}^{\mathrm{Net}}_{\mathrm{U/W}}$ is the SCR for underwriting risk according to the standard formula disregarding the risk mitigating effect for the reinsurance contract
- Collateral is the collateral covering the loss in relation to the counterparty

The 0.5 factor takes into account that the defaulter always will be able to meet a large part of its obligations.

[5] risk reduction

The estimated loss given default in derivative contracts is given by

$$\mathrm{LGD} = 0.5 \cdot \max(\text{Market value} + \mathrm{SCR}_{\mathrm{Mkt}}^{\mathrm{Gross}} - \mathrm{SCR}_{\mathrm{Mkt}}^{\mathrm{Net}} - \text{collateral}, 0) \tag{3.10}$$

where

- Market value is the market value of the financial derivative
- The $\mathrm{SCR}_{\mathrm{Mkt}}^{\mathrm{Gross}}$ and $\mathrm{SCR}_{\mathrm{Mkt}}^{\mathrm{Net}}$ are the SCRs according to the standard formula with and without regarding the risk mitigating effect from the derivatives of the counterparty

The counterparty default risk requirement Def_i for exposure i is based on the Vasicek distribution and is defined by

$$\mathrm{Def}_i = LGD_i \cdot N\left[(1-R)^{-0.5} \cdot G(PD_i) + \sqrt{\frac{R}{1+R}} \cdot G(0.995)\right] \tag{3.11}$$

depending on the implicit correlation R

$$R = 0.5 + 0.5 \cdot \frac{\sum_{i \in Re} LGD_i^2}{(\sum_{i \in Re} LGD_i)^2} \tag{3.12}$$

N is the cumulative distribution function for the standard normal random variable, and G the inverse of N. PD_i is the probability of default for counterparty i. The capital charge of the default risk, $\mathrm{SRC}_{\mathrm{Default}}$, is equal to the sum of the Def_is. Then

$$\mathrm{SCR}_{\mathrm{Default}} = \sum_i \mathrm{Def}_i \tag{3.13}$$

3.7.3 Life underwriting risk

The life underwriting risk includes mortality risk, longevity risk, and disability risk. It also includes lapse risk which is the risk of termination of policies, as well as revision risk and catastrophe risk. The underwriting risk modules are stress tested and summed up, such that all the worst case scenarios from the sub-module risks happen at the same time.

Mortality risk reflects the uncertainty in trends and parameters in the current mortality rates that are not already included in the valuation of technical provisions. It calculates the ΔNAV given an increase in the mortality rate of 10% for each year. The mortality risk will then be stress tested by an unfortunate event. To get the capital charge of the mortality risk we sum up the ΔNAVs over all the policies where the payment of benefits is dependent on the mortality risk.

Longevity risk is the risk that the population lives longer than expected. It calculates the ΔNAV given a decrease in mortality rates of 25% for each year. To find the capital charge we sum up the ΔNAVs over all policies where benefits depend on longevity risk.

Disability risk reflects the risk of uncertainty in parameters in disability rates that are not picked up in the technical provisions. It is calculated by

$$\text{Life}_{\text{dis}} = \sum_i \Delta NAV|\text{disability shock} \tag{3.14}$$

where the disability shock is given by an increase of 35% in disability rates for the next year together with a 25% increase in disability rates at all ages in the following years.

Lapse risk is the risk of change in the insurance liability which occur if the policyholders terminate their policies or don't pay their premium.

$$\text{Life}_{lapse} = \max(\text{Lapse}_{\text{up}}, \text{Lapse}_{\text{down}}, \text{Lapse}_{\text{mass}}) \tag{3.15}$$

where $\text{Lapse}_{\text{Up/Down/Mass}}$ is defined as

$$\text{Lapse}_{\text{up/down/mass}} = \sum_i \Delta NAV|\text{lapseshock}_{\text{up/down/mass}}) \tag{3.16}$$

The $\text{lapseshock}_{\text{up/down}}$ corresponds to an increase or reduction of 50% of the assumed rates of termination in all future years for policies where the termination strain, the difference between the amount currently payable on termination and the expected amount of provision held, is expected to be positive or negative respectively. The $\text{lapseshock}_{\text{mass}}$ is defined as 30% of the sum of surrender strains over the policies where the surrender strains are positive. The result reflects the loss which is incurred in a mass lapse event.

Expense risk reflect the variation in expenses incurred in servicing the insurance and reinsurance contracts. The capital charge of the expense risk is

$$\text{Life}_{\text{exp}} = \Delta NAV|\text{expense shock} \tag{3.17}$$

The expense shock corresponds to a 10% increase in future expenses compared to our best estimate and a 1% increase per annum of the expense inflation rate compared to anticipation.

Revision risk is meant to capture the risk of adverse variation of an annuity's amount as a result of an unanticipated revision of the claims process. Life_{Rev} is defined as ΔNAV given an increase of 3% in the annual amount payable for annuities exposed to revision risk.

Catastrophe risk tries to capture extreme and irregular events which are not captured in any of the other underwriting sub-modules. Life_{Cat} is defined as ΔNAV given that two events occur simultaneously. The first being a 1.5 per mille increase in the rate of policyholders dying within the following year. The second being a 1.5 per mille increase in policyholders experiencing morbidity over the following year.

Now that all the sub-modules of life underwriting risk are explained we define the capital charge for life underwriting risk

$$\mathrm{SRC}_{\mathrm{Life}} = \sqrt{\sum_{r\times c} \mathrm{CorrLife}_{r\times c} \cdot \mathrm{Life}_r \cdot \mathrm{Life}_c} \tag{3.18}$$

where Life_r and Life_c corresponds to the capital charges for the sub-models of life underwriting risk described above according to the rows and columns of the correlation table $\mathrm{CorrLife}_{rxc}$ from table 3.3.

CorrLife	$\mathrm{Life}_{\mathrm{mort}}$	$\mathrm{Life}_{\mathrm{long}}$	$\mathrm{Life}_{\mathrm{dis}}$	$\mathrm{Life}_{\mathrm{lapse}}$	$\mathrm{Life}_{\mathrm{exp}}$	$\mathrm{Life}_{\mathrm{rev}}$	$\mathrm{Life}_{\mathrm{CAT}}$
$\mathrm{Life}_{\mathrm{mort}}$	1						
$\mathrm{Life}_{\mathrm{long}}$	-0.25	1					
$\mathrm{Life}_{\mathrm{dis}}$	0.5	0	1				
$\mathrm{Life}_{\mathrm{lapse}}$	0	0.25	0	1			
$\mathrm{Life}_{\mathrm{exp}}$	0.25	0.25	0.5	0.5	1		
$\mathrm{Life}_{\mathrm{rev}}$	0	0.25	0	0	0.25	1	
$\mathrm{Life}_{\mathrm{cat}}$	0	0	0	0	0	0	1

Table 3.3: Correlation table for life underwriting risk

3.8 Internal model

Undertakings have the possibility of using their own internal model for the calculation of the solvency capital requirements instead of the standard formula. This can either be done through a partial or a full model. The internal model must, however, be calibrated so that the confidence level objective of a 99.5% VaR over one year is fulfilled. It is not required to implement an internal model but undertakings are encouraged to do so since it will give them even better risk assessment, and might reduce the capital requirement. The internal models must be approved by the local FSA before implementation.

3.9 Warning system

A Solvency II goal is to establish an early warning system, and allow more time for supervisory intervention. Table 3.4 represents an adequate ladder of intervention for the two capital requirements:

	Additional Reporting	Financial Recovery Plan	Closure to new business	Authorization withdrawn
No Breach (Adequate Capital)	Not Required	Not Required	Not Required	Not Required
Breach of Adjusted SCR	Required	Possible	Not Required	Not Required
Breach of SCR	Required	Required	Possible	Not Required
Breach of MCR	Required	Required	Required	Required

Table 3.4: Supervisory intervention

The adjusted SCR is defined as the SCR plus any capital add-ons to account for risks which is not fully accounted for in the SCR. These add-ons can be imposed by the supervisor and results in a higher capital requirement. This is to cover the deficiencies in the risk profile of an undertakings business when calculating the SCR.

If a company's capital breaches the SCR requirement the company is forced to improve their financial strength and/or reduce the risk within the portfolio. If the capital is under the MCR "ultimate supervisory action" will be triggered. This means that there will be made plans to transfer the insurers liabilities to another company and the license of the insurer will be withdrawn. In between these two requirements there exist steps which represent degrees of supervisory interventions.

If an insurer is above the Solvency I requirement, but under the Solvency II requirement, they will have one year to comply with the MCR. This means by December 31. 2013.

3.10 Own funds

The main change from previous directives concerning solvency is that the free assets now have to be of good quality as defined in the directive. The directive classifies the quality of own funds into three tiers. They are classified as to how well and how fast they absorb losses.

3.10.1 Tier 1

Tier 1 items are the funds that are available to fully absorb losses on an on-going basis, as well as in the case of winding-up. Winding-up means to conclude business. It entails selling all the assets of a business entity, paying off the creditors, and distributing any remaining assets to the principals, before dissolving the business. For inclusion in Tier 1 the fund has to have the ability to be written down or converted into equity in times of stress.

3.10.2 Tier 2

Tier 2 items are the funds that in case of a winding-up, the total amount of the item is available to absorb losses and the repayment of the item can be refused to its owner until all other obligations have been met. To be able to be classified as Tier 2 any payment from financial instrument has to be able to be deferred in times of stress until financial position is restored.

3.10.3 Tier 3

Tier 3 items shall contain all funds which did not fall into the two other tiers.

In addition to these classifications funds shall be evaluated by their relative duration as compared to the duration of the insurance obligations of the undertaking before deciding which tier they belong to. Duration is a measure of the sensitivity of the price (the value of principal) of a fixed-income investment to a change in interest rates.

3.11 Eligibility

To comply with the solvency capital requirement the eligibility of the funds are subject to quantitative limits. There are two types of eligible own funds which together are the total amount of eligible own funds. The first type comprise the economic capital i.e. the excess of assets over liabilities. This is the eligible basic own funds. The other type comprise the commitments that undertakings can call upon in order to increase their financial resources i.e. letters of credit. This is the ancillary own funds.

Firstly the proportion of Tier 1 items shall be higher than one third of the total amount of eligible own funds. Secondly the proportion of Tier 3 items shall be less than one third of the total amount of eligible own funds.

As far as the minimum capital requirement is concerned there are quantitative limits as well. Here the ancillary own funds are not eligible. The proportion of Tier 1 items must be higher than one half of the total amount of eligible basic own funds.

The eligible amounts of own funds to cover the SCR is equal to the sum of the amount of Tier 1 and the eligible amounts of Tier 2 and Tier 3. The eligible basic own funds to cover the MCR should be equal the sum of the amount of Tier 1, and the eligible amounts of basic own fund items classified as Tier 2.

3.12 Solvency requirement figure

To ensure that we have enough assets, and to comply with the SCR of the Solvency II directive, we calculate the VaR of insurance and financial liabilities with a 99.5% confidence level. In addition to the technical provisions, an insurer will need to hold the value of the SCR in good quality assets to comply with the directive. An illustration of the different sizes mentioned in this chapter is given in figure 3.2.

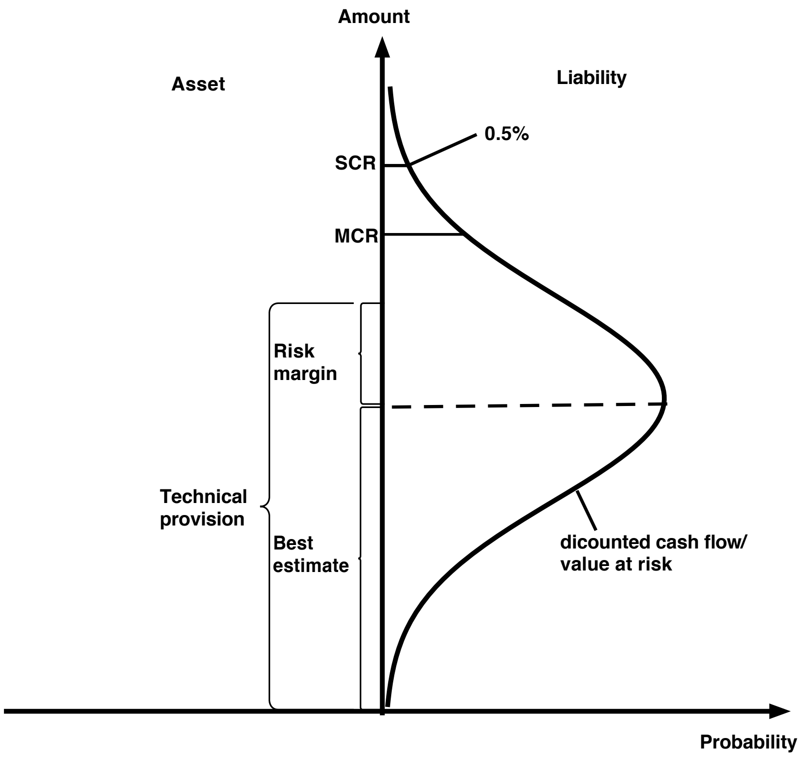

Figure 3.2: Illustration of solvency requirements

3.13 Conclusion

The Solvency II directive answers all the big questions regarding solvency. It defines how large the solvency requirement should be, and how to calculate it. The time horizon to considers regarding the solvency requirements is one year. The directive also states which assets are eligible to cover the solvency requirement.

There is much work to be done with the implementation of the new directive. The final parameters for the stress test and correlation matrices need to be set, and the final quantitative impact study, QIS 5, has to be held.

The directive has concentrated on improving the overall risk assessment of the companies. The choice of the correlation parameters is however difficult to understand. It is not explained how these parameters are chosen, only that they presumably give the wanted 99.5% VaR measure. The MCR calculation is also a little uncertain. As of now it is a combination of two different approaches. I assume they will have to choose one of them, and preferably the risk based alternative.

For the undertakings there are a lot of calculations to be done in order to find the new solvency requirements. The undertakings will be required to report their calculations to the supervisory authorities. This will involve new projects for the insurance undertakings. Even more work if they decide to implement an internal model. However, if the directive achieves its goals, it will help the undertakings to have control over their financial situation and risks. It will also give the regulatory authorities the necessary data for supervising the solvency of the insurance undertakings.

In the next chapter I will study the expected changes for the Norwegian life insurance undertakings, and evaluate the QIS 4 study that has been run for these undertakings. I will then continue by studying the valuation aspect of the directive. To be able to calculate the solvency capital according to the Solvency II directive requirement we need a "full balance sheet" approach. This means to measure both the assets and liabilities in one consistent way. In order to do so we will use market values because of its switching property. By using market values we can make arrangements to switch assets to cover our liabilities at market price. It is therefore clear that we need a market-consistent valuation, also called mark-to-market model, of the balance sheet. I will eventually show how to construct a valuation portfolio that reflects a market-consistent actuarial valuation.

4

CONSEQUENCES FOR NORWEGIAN UNDERTAKINGS

The Solvency II working progress follows the Lamfalussy process, which is an approach to the development of financial service industry regulation used by the EU. It is composed of four levels, each focusing on a specific stage of the implementation of the legislative process.

At the first level a legislation is adopted by the European Parliament and the Council of the EU. The core values and building guidelines of the implementation is set. At the second level supplementary sector-specific committees and regulators advise on technical details. At the third level the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS) develop harmonized standards across the member states. CEIOPS consists of the European Union's insurance and pension fund supervisory authorities. The third level is where the Quantitative Impact Studies (QIS) are carried out, see 4.1. At the fourth level, the new directive will be implemented.

By evaluating the QIS results for the Norwegian undertakings we can comment on a few key modifications.

4.1 Quantitative Impact Studies

The European Commission is responsible for outlining the solvency directive. They have received technical advice from the CEIOPS on legislative matters. CEIOPS has in this regard supervised simulations on the preliminary proposal of the Solvency II directive. These simulations are called quantitative impact studies. The QIS are a way of testing out the parameters involved in the standard formula and the new requirements for financial capital. The QIS is conducted by insurance undertakings, and are done on a voluntary basis throughout the EU. Up until now there have been conducted four such studies, and the fifth will be conducted during the fall of 2010. The studies will ensure that the final SCR formulas and parameters are calibrated correctly, such that the capital requirement's VaR is set at a confidence level of 99.5%. The final parameters will be decided by the end of 2010. Then the companies can begin implementing the Solvency II directive. The QIS 4 simulations used data as of December 31. 2007. The simulations were run between April-July of 2008.

4.1.1 Norwegian result for life insurance undertakings

Results from QIS 4 show that 40% of the Norwegian life insurance companies participated. These companies have 92.3% of the market shares of participants. From the study it turns out that under the new Solvency II requirements, the mean available capital to SCR is 112%. Compared to the Solvency I requirements the mean available solvency margin is 198%. Although these percentages are not directly comparable (see figure 3.1), it gives us a hint that the new directive requires more capital to avoid supervisory interference. It implies that under the new directive companies might need to bring in more capital, or take less risk in the form of selling out part of their stock portfolio and buying bonds instead.

The market risk component is the most dominant risk in the SCR calculation with an average of 87% of the BSCR^* where

$$\mathrm{BSCR}^* = \mathrm{BSCR} + \mathrm{SCR}_{\mathrm{OP}} \tag{4.1}$$

In comparison the life underwriting risk accounts for an average of 28% of the BSCR^*. The counterparty risk is negligible with 0.2%, and the capital charge of operational risk amounts to 2.1% of the BSCR^*. The diversification effect due to correlation assumptions between the risks mentioned is 17%.

The adjustment factor in the SCR calculation, see equation (3.3), is stipulated to be more than 45% of the BSCR^* on average. There is an ongoing discussion concerning the different approaches being used to valuate future discretionary dividends to policyholders before and after stress. Therefore we observed variations in the adjustment factor from 35-55%. Further specification in the implementing measures will have a significant impact on the overall SCR.

Finanstilsynet considers making the participation in the QIS 5 mandatory for Norwegian insurance undertakings. None of the Norwegian insurance companies reported SCR calculations based on an internal model.

4.2 Key modifications for Norwegian undertakings

The Solvency II directive will go through the Norwegian legislation by October 2012. This implies that the new directive is to be applied for the fiscal year 2013. The preparation to follow the Solvency II directive will begin simultaneously with the implementation of the new pension reform which is to be implemented in 2011. If the insurance undertakings want to use an internal model, the preparation should begin even sooner. It will lead to a lot of extra work in order to implement both the directive and the reform, and undertakings should be prepared for this.

4.2.1 Governance and control

Companies will be required to carry out its Own Risk and Solvency Assessment (ORSA) under the Solvency II directive which will be an integrated part of the operation of the company. The ORSA is an internal assessment process within the undertaking. It is also a supervisory tool for the supervisory authorities. The aim is to identify whether the particular risk profile of an undertaking deviates from the assumptions underlying the standard or internal model.

4.2.2 Valuation

Assets and liabilities shall be valuated using "realistic" values. This will lead to a change in how the insurance liabilities is valued i.e. best estimate element plus a risk margin. On the asset side, the held-to-yield bonds shall furthermore be valuated by its real value. For accounting purposes the value of held-to-yield bonds are as of now determined by the price it was bought for, the time to maturity, and its return. Under the new solvency directive these bonds must be valued using the market price.

Maybe the most important change with the new directive is the use of a risk-free rate of interest for discounting. As of now the discounted technical provisions are calculated by using a fixed technical rate determined by Finanstilsynet. Switching to a risk-free interest rate implies that the technical provisions will fluctuate with the interest rate.

4.2.3 Calculations of capital requirements

As stated above there are two new capital requirements, the SCR and MCR which have to be calculated. There is also the possibility of developing full, or partly internal models for the capital requirement. This model must be based on known risk measures, and be calibrated with an equal VaR as the standard formula. It has to be validated, tested, and documented. The insurance companies should have begun developing such models by the beginning of 2010 to be able to implement it at the same time as the rest of the new directive. An internal model will have to be approved by Finanstilsynet before it can be used to calculate the capital requirement instead of the standard formula.

4.2.4 Own Funds

The quality of the capital is broken down into three tiers as described in section 3.10. Assets that entered in the required solvency margin of the Solvency I directive does not necessarily enter into the solvency capital requirement of the Solvency II directive. Stricter requirements to the quality of the assets might imply that the undertakings will need to shift around their assets to meet the new requirements. It may also mean that they need to bring in more capital to their asset portfolio.

4.2.5 Reporting

There will be new requirements for reporting to Finanstilsynet and the public both in format and frequency. This includes a change in the existing reports such as actuarial reports, and solvency margin reports. As of June 2009 Finanstilsynet requires the insurance undertakings to perform a stress test quite similar to the standard formula calculations of the Solvency II directive. These stress tests are to be handed in each quarter.

5

THE BEST ESTIMATE CASH FLOW

In section 3.5 we defined an insurers technical provisions by the capital the insurer would need to meet his obligations towards the policyholders. We also insisted that the technical provisions equal the discounted best estimate of the companys cash flows plus a risk margin. In this section we shall describe the cash flows involved in the insurance industry, and how they are calculated. I will only consider cash flows which occur in life insurance, and disregard the cash flows involved in the non-life insurance industry. In life insurance the undertakings are liable either if the insured dies, or if he becomes disabled. If the insured dies, there might be a spouse, a partner, a cohabitant or children that are entitled to compensation. The compensation can either be a one-time payment, or in monthly or yearly payments for a specific time period or until death.

5.1 Basic model

In order to define a model for calculating the best estimate cash flows I will discuss the theory behind the basic discrete time model for a random cash flow. This model will include concepts from probability theory. We assume we have a probability space $(\Omega, \mathcal{F}, P)$ and an increasing sequence of σ-fields $(\mathcal{F}_t)_{t=0,\dots,n}$ with

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \mathcal{F} \tag{5.1}$$

- Here $\emptyset$ denotes the empty set, Ω is the sample space that contains all possible outcomes of the random experiment, $\mathcal{F}$ is a σ-field, and $P(\cdot) : \mathcal{F} \to [0, 1]$ is a probability measure.
- In the present case of an insurers' random cash flow, $\mathrm{P}(\omega) > 0$ for all $\omega \in \Omega$. Also $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$ that describes how information about payments are revealed to the insurer.

- We will think of the cash flow that the insurer faces as a sequence of $\mathcal{F}_t$ adapted random variables $X_0, X_1, \ldots, X_n$. Also, in this chapter, we will let X denote a sequence of cash flows with $X_i = 0$ for i $\neq$ t, and with a single payment of X_t at time t.

In this chapter, our main goal will be to find the value of the cash flow X.

5.2 Cash flows

The best estimate cash flow from the Solvency II directive is a cash flow **X**, which consists of several seperate cash flows. Let us first consider a policy in one specific time period. The policyholders pay a premium, Π_t at time t. This premium is calculated and charged by the insurer and corresponds to the policyholders expected financial expenses and benefits which the insurer is liable for. The insurer then pays the policyholders for actually incurred expenses and benefits, C_t, within the time period (t - 1,t]. If we assume that all the payouts occur at the end of the time period, the discrete cash flow, X, at time t equals

$$X_t = -\Pi_t + C_t \tag{5.2}$$

It is not a requirement for the X_t to be negative at all times. However it is a requirement that X_t is measurable with respect to $\mathcal{F}_t$. Due to the fact that not all the liabilities are settled at once, it is possible to generate income from investments on the premiums and delayed settlements on the condition that you are able to meet your liabilities at all times.

5.3 The best estimate cash flow model

It is impossible to know exactly how much an insurance undertaking is liable for in claims and expenses each year. This is due to an unknown mortality and disability rate. Because of this uncertainty we are satisfied by finding a best estimate for our insurance liability cash flow. I will describe the calculation of this cash flow from the QIS 4's technical provisions definition.

> The best estimate is equal to the expected present value of all future potential cash flows (probability weighted average of distributional outcomes), based upon current and credible information, having due regard to all available information and reflecting the characteristics of the underlying (re)insurance portfolio.

I will discuss the interest rate used to obtain the discounted best estimate cash flow in the next sections, but first I want to find a realistic expression for the best estimate of the cash flow at a given period t. To find the estimate we would begin by analyzing historical data for disability or mortality rates, using statistical methods. Then we would need to find a distribution with parameters such that the incurred data for disability or mortality rates fit. Fortunately this work has already been done for us. For the Norwegian population parameters from either K2005 or GAP07 can be used. They are prepared by FNH[1] and Gabler Wassum AS respectively. They both offer mortality rates as well as the probability of surviving spouse and children, and disability rates for both men and women.

I will elaborate a bit on the mortality rate. The FNH parameters, which probably are the most used parameters in Norway, use a Gompertz-Makeham mortality model with a mortality rate equal to the sum of a constant, and an age-dependent component which increase exponentially with age:

$$q_x = 1 - \exp(\theta_0 + \theta_1 e^{\theta_2 x}), \tag{5.3}$$

where q_x is the probability of death within one year given age x. The Gompertz-Makeham model is pretty accurate for ages up to 100. For advanced ages however, it actually predicts a too high mortality. The GAP07 therefore includes additional parameters in order to compensate for the inaccuracies outside the given interval. The GAP07 mortality rate

$$q_x = \frac{1}{1 + e^{h_x}} \text{ , where } h_x = \theta_0 - \frac{\theta_2}{1 + e^{-\theta_1 \cdot S(x - x_0)}} \tag{5.4}$$

and

$$S(x) = \begin{cases} 0.01 \cdot x, & \text{if } x \leq 0 \\ (0.01 \cdot x)^{\gamma}, & \text{otherwise} \end{cases} \tag{5.5}$$

A figure of the Norwegian mortality for men calibrated for the year 2004 is given in figure 5.1. The equations (5.3) and (5.4) are used. The parameters are found in [3]. For the Gompertz-Makeham rate they are $\theta = (-0.00309, -0.0000219, 0.100047)$. And for GAP07 they are $\theta = (9.03984, 3.50737, 12.35429)$, $x_0 = 78$ and $\gamma = 0.93969$.

It is important not to use the parameters uncritically, due to several sources of error which we have to be aware of. The choice of parameters might not be the optimal in the future, even though it was so in the past. This can cause estimation error. One issue is that the population seems to live longer and longer. This should be picked up by the new rates. Another problem is that the insured population might have a slightly higher life expectancy than the population as a whole. Reasons for this could be

- Regional: there could be regional differences in mortality. There might not be the same types of jobs available in the countryside as in the big cities. Traffic accidents have a higher intensity in the cities.
- Ethnic: the population might include a higher number of immigrants than the population as a whole. This group might have different life expectancies.
- Occupational: specific occupational groups are more physically demanding which could cause higher disability rates. Other groups are more exposed to accidents. And office jobs might be related to a higher life expectancy.

[1] Finansnæringens Hovedorganisasjon

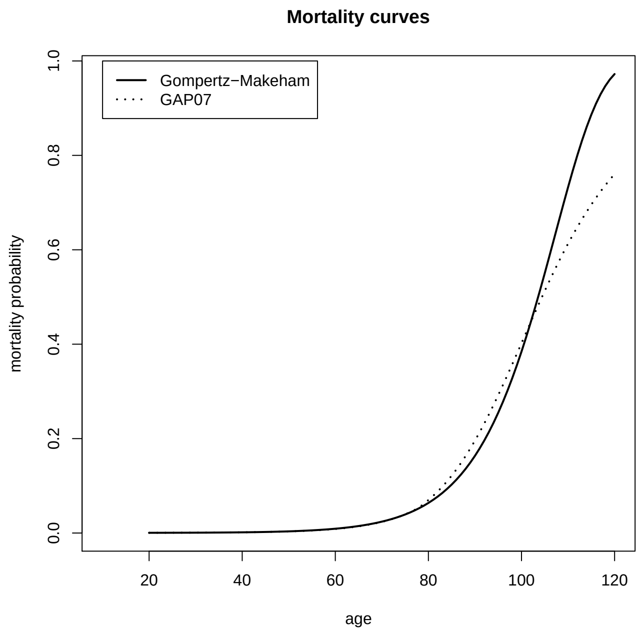

Figure 5.1: Gompertz-Makeham and GAP07 mortality curves

All of these factors could cause a drastically higher liability estimate. The errors mentioned above are in general called technical risk. There is a need to calibrate the mortatility rates to our insurance portfolio, and we are left with an estimate for our best estimate which is the expected value of expenses the company is liable for.

The expected best estimate cash flow will only be a prognosis of the real value because of the technical risk, and we will therefore always have deviations from the true outcome. In other words, if all parameters involved in the cash flow estimation could be determined with absolute certainty, and the parties involved were risk-neutral, then the best estimate value would equal the market value.

Our best estimate element described above needs to be discounted to find the best estimate as stated in the Solvency II directive. It has been common practice in Norway to discount the best estimate with a technical rate of interest, but when calculating the market value of our liability cash flow it would be wise to use a market rate of interest.

5.3.1 Technical rate

The Solvency II directive states that the best estimate cash flow is to be discounted using a risk-free interest rate. As of now however, most contracts in Norway contain an interest rate guarantee which the Norwegian regulators, Finanstilsynet, have defined.

The rate has been chosen with respect to the EU life insurance directive[2]. The directive states that the highest rate of interest should be determined with caution and not exceed 60% of the long-term government treasury bill rates. In Norway the longest government bonds have a duration of 10 years. Taking the directive into account the maximum rate is set to be no higher than 2.75%. This implies that the discount rate, ϕ_t, is set to a constant rate

$$\phi_t = \frac{1}{(1+0.0275)^t} \tag{5.6}$$

at time t. Setting the rate of interest lower than the maximum rate will lead to increased premium and lower interest rate guarantee for policyholders. Although the insurance companies by law are enforced to use a constant discount rate when calculating premiums and premium reserves it would be wise to calculate outcomes with a stochastic discount rate as well. The advantage of using a constant discount rate is that it gives a consistent and predictable theory, but it is not necessarily realistic compared to observed rates. When converting to the Solvency II regime we should obtain rates that are consistent with the financial market, i.e. the risk-free interest rate.

5.4 Discounting under Solvency II

QIS 4 states that all cash flows should be discounted using the risk-free discount rate applicable for the relevant maturity at the valuation date. Where the market does not provide data for maturity, extrapolation or interpolation shall be used. In QIS4 the risk-free interest rate was derived from swap rates. If there did not exist a swap rate market, government bonds were to be used. Where cash flows were linked to financial instruments, the market values of the financial instruments were to be used. Hence, an equation of the discounted value of the best estimate cash flow ($\phi \cdot \mathbf{X}$) at time t under the real world probability measure P will look like

$$E(X_t) = E(\phi_t \cdot \mathbf{X} | \mathcal{F}_t) \tag{5.7}$$

where we condition on the information revealed at time t. I will discuss the use of zero coupon bonds and swap rates as discount factors in chapter 6.

5.5 Conclusion

We have described a way of calculating the best estimate cash flow. We have also discussed which interest rate we should use to discount our best estimate of an insurer's cash flow during a given period t. Under the new Solvency II directive, we shall make use of the risk-free interest rate. We will discuss this stochastic discount rate further in chapter 6. When discounting the best estimate cash flow, we shall make use of the

[2]Directive 2002/83/EC Article 20 No.1, B.(i)

σ-field $\mathcal{F}$ that we described at the beginning of the chapter. The cash flow information is revealed in accordance with $\mathcal{F}_t$.

6

STOCHASTIC DISCOUNTING

Now we know how the best estimate of the cash flow pertaining to a given period is calculated. For accounting purposes and in exchange with the regulator we want to be able to discount our best estimate cash flow. The QIS 4 specifies different alternatives which can be applied. We want to find a discount factor, or numeraire, which transforms our cash flow into a fair value, which implies an arbitrage free valuation.

6.1 Arbitrage-free valuation

In this chapter we consider the same insurance company that we discussed in chapter 5. In particular, we assume that the company's best estimate cash flows can be represented by random variables on a probability space $(\Omega, \mathcal{F}, P)$, that are adapted to an increasing sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$ of sub-σ-fields of $\mathcal{F}$. In chapter 5 the best estimate cash flow, X, was an (n+1)-dimensional vector of the form $(0, \ldots, 0, X_t, 0, \ldots, 0)$. In this chapter $X = (X_0, \ldots, X_n)$. To handle such cash flows, we assume that we have a function $Q_t : L^2_{n+1}(P) \to \mathbb{R}$ which is a positive, continuous, linear function in L^2_{n+1}(P). We then define a mapping of the best estimate cash flow $\mathbf{X} \mapsto Q_t(\mathbf{X})$, which represents the discounted monetary market value $Q_t(X) \mapsto \mathbb{R}$ of the cash flow $\mathbf{X}$ at time t.

We wish to consider a market without arbitrage opportunities. This is a market where there does not exist a risk-less way of making money. Up until now we have operated with a "real world"-probability measure, P. We need to introduce the definition of martingales, and a risk neutral probability measure P^*. The fundamental no arbitrage theorem states that there are no arbitrage opportunities if and only if there exists a risk-neutral probability measure.

Let us begin by defining *martingales*. Let $Y_t, t = 0, 1, \ldots, n$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n$ be the given increasing sequence of sub-σ-fields of $\mathcal{F}$. Then Y_t is a martingale if

- Y_t is $\mathcal{F}_t$-adapted
- $\mathrm{E}[|Y_t|] < \infty$ for all $0 \leq t \leq n$
- $\mathrm{E}[Y_t|\mathcal{F}_s] = Y_s$ for all $0 \leq s \leq t$
- The Brownian motion is a martingale. $\mathrm{E}[B_t|\mathcal{F}_s] = B_s$ for all $0 \leq s \leq t$

A Brownian motion is a stochastic process on $(\Omega, \mathcal{F}, \mathrm{P})$ where

- $B_0 = 0$
- the increments are independent such that $B_{t_2} - B_{t_1}$ is independent of $B_{t_3} - B_{t_2}$ for $0 < t_1 < t_2 < t_3$.
- the increments follow a normal distribution, i.e. $B_{t_2} - B_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$

Now let us define a risk neutral probability measure for our insurer's cash flows on $(\Omega, \mathcal{F}, P)$. Suppose that

- P^* is a probability measure with strictly positive mass to every state $Q(\omega) \in \Omega$
- the discounted cash flow price $(\phi_t \cdot X_t)$ is a martingale under P^* for t $\in (0, \ldots, n)$.

Then P^* is a risk neutral probability measure, also called a martingale measure. If we can prove that there exist a P^*, then we have found a risk-neutral valuation of our price process. We will see that in a risk neutral world the expected price of a process evolve according to $\mathcal{F}_t$.

6.2 Discounting

The price process of our best estimate cash flow, $Q_t(\mathbf{X})$, can be expressed as an expectation on the sum of the cash flow multiplied by a stochastic discount factor, ϕ, which is called a deflator or numeraire.

By Riesz representation theorem there exists a vector $\phi \in L^2_{n+1}(P)$ s.t. for all $\mathbf{X}$ in $L^2_{n+1}(P)$ equation (6.1) holds.

$$Q_t(X) = E\left(\sum_{k=0}^{n} X_k \cdot \phi_k | \mathcal{F}_t\right) \tag{6.1}$$

ϕ is a stochastic discount factor. Be aware of the fact that we do not know a general expression for the value of the discount factor, the theorem only states that a discount factor exists.

Let us look at two interesting properties of the discount factor. The first property is that the positivity of the price process Q ensures positivity of ϕ. The other property, which will be very useful in further calculation, is that we can choose ϕ_t to be $\mathcal{F}_t$-adapted. Consider the following calculation in 6.2 where we replace ϕ by $\tilde{\phi}_k = E(\phi_k|\mathcal{F}_k)$.

$$\begin{aligned} Q_t(\mathbf{X}) &= E\left(\sum_{k=0}^{n} X_k \cdot \phi_k|\mathcal{F}_t\right) = E\left(\sum_{k=0}^{n} X_k \cdot E(\phi_k|\mathcal{F}_k)|\mathcal{F}_t\right) \\ &= E\left(\sum_{k=0}^{n} X_k \cdot \tilde{\phi}_k|\mathcal{F}_t\right) \end{aligned} \tag{6.2}$$

Equation (6.2) holds for all $\mathbf{X}$. By this reasoning we shall from now on assume that ϕ_t as well as $\tilde{\phi}_t$ is $\mathcal{F}_t$-adapted. This means that we can construct a discount factor where we know its value at time t. With ϕ_t we have found a direct link between the behavior of the financial market and the $\mathcal{F}_t$-adapted cash flow X_t. We shall assume that $\phi_0 = 1$.

6.3 Risk-free interest rate

A risk-free interest rate is an interest rate which can be obtained by investing in a financial instrument without default risk. There does not exist truly risk-free interest rates, but we can find pretty good estimates. A fair value estimate uses market expectations on future interest rates. An estimate could be short-term government bond rates like US treasury bills. LIBOR[1] and EURIBOR[2] can also be used. A treasury bill can in theory not default, because the government can print new bills to pay its debts. This will however lead to devaluation of the currency used. LIBOR rates are the approximated rates for a short-term borrowing rate of an AAA credit rated[3] company. An AAA-rated company has a default risk of 0.01%. The discount rates can be represented by

$$\phi_t = \frac{1}{(1+\bar{r}_0)^t} \tag{6.3}$$

where $\bar{r}_0$ corresponds to the market interest rate i.e. the interest rate observed in the market.

We need to be aware of two main risks that can affect a bond's investment value. The first is liquidity risk which is the risk of being unable to sell the instrument for cash at short notice without significant costs. The other is market risk which includes interest rate risk, equity risk and currency risk.

Another candidate to the risk free interest rate is the swap rate. The swap rate derives from an interest rate swap which is an agreement between two counterparties where one

[1] London Interbank Offer Rate

[2] European Interbank Offer Rate

[3] Refers to the rating agency Standard & Poor's best credit rating. The second best being AA. AAA correspond to Aaa from the rival rating agency Moody.

agrees to pay a fixed rate (the swap rate) and the other agrees to pay a floating rate. The rates are pinned to a notional principal amount to decide the size of the cash flow which will be exchanged between the counterparties. The floating rate to be used under the QIS 4 specifications is the 6-month EURIBOR. The interest rate swap is constructed so that no initial payment is made. This makes the market value of the swap equal to zero. Equation (6.3) still holds where $\bar{r}_0$ now is defined as the swap rate.

According to QIS 4, further work will need to be conducted to see whether swap rates are an appropriate benchmark to determine the risk-free interest rate term structure, once liquidity considerations have been taken into account. But the swap rate has however been used in the simulation studies carried out so far.

Our problem with using the market interest rate, $\bar{r}_0$ is that the rate is only known up to a certain time. After that we need models to predict the real interest rates and bond prices. Here

$$\text{real interest rate} = \text{nominal interest rate} - \text{inflation rate} \tag{6.4}$$

The Black-Karisinsky model, the Vasiček model, or the Cox-Ingersoll-Ross can be used to estimate the interest rate, see Lamberton and Lapeyre [10]

6.4 Risk neutral discounting

We let our discount factor in a discrete world follow equation (6.3). We have in other words discounted the price process with a risk free interest rate. At time t the price process Q_t is given by conditioning on the σ-algebra $\mathcal{F}$

$$Q_t(\mathbf{X}) = Q(\mathbf{X}|\mathcal{F}_t) = E_{P^*}\left(\sum_{k=0}^{n} X_k \cdot \phi_k|\mathcal{F}_t\right) \tag{6.5}$$

In order for the price process to be risk neutral we have switched to the risk neutral probability measure, P^* such that

$$P^*(A) = E(\phi \cdot 1_A) \tag{6.6}$$

6.5 Equivalent martingale measure

Sometimes it is hard to find the statistical distribution of the underlying process you are evaluating. Therefore it is possible to change the probability measure. The process might fluctuate with for example a bond rate. Then by dividing by a numeraire P(0,T) we obtain a new equivalent probability measure.

$$\tilde{P}(A) = \frac{1}{P(0,T)} E_{P^*}(\phi \cdot 1_A) \tag{6.7}$$

This transformation may ease computations of prices of interest rate derivates. In order to maintain the risk neutral quality, the price process has to meet the risk neutral requirements mentioned before, but now under the new transformed martingale measure. When found, this measure is called an equivalent martingale measure. This measure operates in a new risk neutral world.

6.6 LIBOR forward rate evolution

An example of a risk-free interest rate that can be used under the Solvency II directive, is the LIBOR forward rate. I will therefore present its evolution in this section.

The problem with the interest rate estimations such as the Vasicěk model is that they give explicit functions of the instantaneous spot interest rate. The LIBOR forward rate, on the other hand, gives a continuous rate that models the evolution of the whole yield curve forward in time. We will first look at the evolution of the LIBOR rate, then we will illustrate how to simulate the LIBOR forward rate using theory introduced in this chapter.

By applying a forward hedging strategy of the LIBOR rate, L(t,T), we have the following expression

$$L(t,T) = \frac{P(t,T)P(t,T+\delta)}{\delta \cdot P(t,T+\delta)}, \tag{6.8}$$

where P(t,T) denotes bonds with maturity T, and δ is the tenor of the LIBOR, i.e. 0.25 or 0.5 years. We want to find an expression for the evolution of the LIBOR forward rate under an appropriate change of measure to ease computations. First we need the Girsanov theorem.

6.6.1 Girsanov's theorem

This theorem roughly says that an adapted process U_t, given by a Brownian motion distorted by a noise remains Brownian under a new risk neutral probability measure P^*. A process is called adapted when it is dependent on a Brownian motion, B_s for $s \leq t$. We let U_t be

$$U_t = B_t - \int_0^t V_s ds \tag{6.9}$$

for an adapted process V_t. Set

$$Z_t = \exp(\int_0^t V_s dB_s - \frac{1}{2}\int_0^t |V_s|^2 ds), \quad 0 \leq t \leq T \tag{6.10}$$

and assume that $E[Z_t] = 1$ for all t. Then Z_t is a martingale under P. The risk neutral probability measure is defined

$$P^*(A) = E[1_A \cdot Z_T] \tag{6.11}$$

and by Girsanov's theorem, the process U_t is a new Brownian motion defined as in (6.9), now under P^*. It is therefore also a martingale under P^*.

6.6.2 T-forward measure

A T-forward measure is a pricing measure absolutely continuous with respect to a risk-neutral measure but rather than using the money market as numeraire, it uses a bond P(0,T) with maturity T. We define the T-forward measure

$$\tilde{P}^T(A) = \frac{1}{P(0,T)} E_{P^*}[1_A \cdot D(t)] \tag{6.12}$$

where D(t) is the continuous discount rate

$$D(t) = e^{-\int_0^t r(s)ds} \tag{6.13}$$

Then

$$\tilde{B}_t^T := \tilde{B}_t - \int_0^t \sigma^*(s,T)ds \tag{6.14}$$

is a Brownian motion under $\tilde{P}^T$ by Girsanov's theorem, and $\sigma(t,T)$ is the risk premium, where $\sigma^*(s,T) = \int_t^T \sigma(t,s)ds$. The latter can be shown with the help of integration by parts, Itô's formula and Girsanov's theorem.

6.6.3 Evolution of L(t,T)

We apply Itô's formula on g(t,X) $= L(0,T)e^X$ where

$$X_t = \int_0^T \gamma(s,T_j)d\tilde{B}_s^{T_{j+1}} - \frac{1}{2}\int_0^T \gamma^2(s,T_j)dt \tag{6.15}$$

for a volatility parameter process

$$\gamma(t,T) = \frac{1+\delta \cdot L(t,T)}{\delta \cdot L(t,T)} \cdot (\sigma^*(t,T+\delta) - \sigma^*(t,T)) \tag{6.16}$$

Then the evolution of the LIBOR forward rate is given by

$$dL(t,T) = \gamma(t,T)L(t,T)d\tilde{B}_t^{T+\delta}, \quad 0 \le t \le T \tag{6.17}$$

where $\tilde{B}_t^{T+\delta}$ is a Brownian motion under $\tilde{P}^{T+\delta}$. Details are supplemented in Appendix A.1.

6.7 LIBOR forward rate simulations

We will estimate the γ with a non random function $\tilde{\gamma}$ which will be derived from caplet volatilities of market data. The forward rate simulation formula will derive from

$$dL(t,T_j) = \tilde{\gamma}(t,T_j)L(t,T_j)d\tilde{B}_t^{T_{j+1}} \tag{6.18}$$

where the relationship $\tilde{B}_t^{T_2},\ldots,\tilde{B}_t^{T_{n+1}}$ must be found. The calculations are tedious but straight forward and are supplemented in Appendix A.2. We end up with the evolution

$$\begin{aligned} L(t,T_j) = L(0,T_j)\exp\{&\int_0^t \tilde{\gamma}(s,T_j)d\tilde{B}_s^{T_{n+1}} \\ &+\int_0^t (\tilde{\gamma}(s,T_j)(-\sum\nolimits_{i=j+1}^{n}\frac{\delta\tilde{\gamma}(s,T_i)}{1+\delta L(s,T_i)}) - \frac{1}{2}(\tilde{\gamma}(s,T_j))^2)ds\} \end{aligned} \tag{6.19}$$

for $0 \le t \le T_j, \quad j = 1,\ldots,n$. Simulations over three different time periods are plottet in Appendix B.

6.8 Conclusion

We have looked at how to discount our best estimate cash flow using an arbitrage free valuation. There are different alternatives to the risk-free interest rate, and we have discussed the use of government bonds, LIBOR and EURIBOR rates, and swap rates. All which lead to a theoretical risk free valuation. The negative side was that they only gave the spot interest rate, and not the whole yield curve. That is why we looked at the LIBOR forward rate, and illustrated how to simulate these forward rates. The theory and the calculations involved were also presented.

There is more work to be done in order to determine the risk-free interest rate structure which shall be used under the Solvency II directive. In the next chapter, where we are going to derive the value of our insurance portfolio, we will not discount as described in this chapter. We will instead link our cash flow to financial instruments. This will enable us to achieve a market consistent valuation of the technical provisions throughout the period, without the use of stochastic discount factors.

7

MARKET CONSISTENT VALUATION OF TECHNICAL PROVISIONS

Now we will look at a new method of valuing an insurance company's cash flow introduced by Bühlmann et al. [17]. It is a method of valuation which does not involve discounting in the same manner as in chapter 6. It provides the possibility of finding the market value of the technical provisions directly, and not only of the discounted best estimate cash flow.

I have earlier stated that the technical provisions are made up of the discounted best estimate cash flow, and a risk margin. The Solvency II directive stated that we are allowed to valuate the two elements together when the future cash flow associated with insurance obligations can be replicated using financial instruments for which market values can be found. If so the value of the technical provisions should be determined by the market value of those instruments that comprise the technical provisions. We are therefore going to valuate the technical provisions as a unit, linked to financial instruments and in that way determine the market value of the insurance portfolio. When this valuation is found we will know how to obtain solvency, either by replicating the portfolio we have found, by hedging, or by setting up a risk bearing capital.

In order to find the market value of the insurance portfolio we need to connect our technical provisions to financial instruments. We will then obtain a market consistent valuation of an insurer's cash flow. The model is risk-adjusted and builds a bridge between the actuaries and the asset managers by looking at the "total balance sheet" instead of looking at the assets and liabilities separate. As of now the actuaries are in charge of the liabilities, and the asset managers are in charge of the active side of the balance sheet. This will most likely change when the new Solvency II directive is implemented. The new directive concentrates both on the asset-side risk as well as the technical risk. Therefore there should be interaction between the asset and liability side. By applying this new model we are able to measure both sides in one consistent way. This should lead to a successful Asset Liability Management (ALM) strategy.

7.1 Market values for the cash flows

Our goal is to find a market consistent model for calculating the technical provisions cash flow. We need to introduce a fair value concept. The definition of fair value is

> the amount which an asset could be exchanged or a liability settled between knowledgeable willing parties in an arm's length transaction

The assets of a company can usually be traded in a market. The fair value for the assets is, therefore, generally taken as equal to the market value. Our problem lies in the fact that the technical provisions cash flow is not usually traded in a market. For calculation we therefore might think of this value from a run-off situation, where the insurance portfolio does not take in new members, but continues its business until the last expenses are paid. In value this is equal to the amount one insurance company would have to pay to another insurance company to transfer its obligations. The Solvency II directive states

> The value of technical provisions should therefore correspond to the amount an insurance or reinsurance undertaking would have to pay if it transferred its contractual rights and obligations immediately to another undertaking

So how should we valuate an insurance portfolio in life insurance? We begin by breaking the procedure down into two separate steps. First we shall assume that the technical provisions cash flow is deterministic. This means we assume the life table and mortality rates are known. The only stochastic contribution will come from the financial instruments we link our estimate to. In the next step we shall assume that we have a stochastic technical provisions cash flow. We will then use the construction from the first step, and add loadings for the technical risks that arise from a stochastic life table. This will in the end give us the construction of a insurance portfolio, protected against technical risk, and in terms of financial instruments. If we are able to find the market value of the financial instruments then we have managed to find a market value for our technical provisions.

We will follow an example of an insurance cash flow, where premium is paid in advance. If the policyholder dies, a benefit is paid to the beneficiary. If the policyholder survives until a predetermined age, he will receive the benefits he has accumulated.

7.2 Deterministic life table

We assume that we have a deterministic life table, where no technical risk is involved. The construction of such a valuation portfolio can be broken down into three steps. First let l_x denote the number of individuals alive in year x, and let d_x denote the number of individuals who are alive in year x, but die before reaching the year $(x+1)$. Then we have the following general diagram:

year	survived		died
x	l_x		
		$\rightarrow$	$d_x = l_x - l_{x+1}$
x+1	l_{x+1}		
		$\rightarrow$	$d_{x+1} = l_{x+1} - l_{x+2}$
x+2	l_{x+2}		
		$\rightarrow$	$d_{x+2} = l_{x+2} - l_{x+3}$
$\vdots$	$\vdots$		$\vdots$

Further we will, as an example, denote the time by $\{t_0, \ldots, t_5\}$.

7.2.1 Step 1

The first step is to determine into which units we are going to decompose the insurer's technical provisions. We will use the units of a multidimensional vector for the valuation. Specifically, we divide the technical provisons cash flow into premium, death benefit, and survival benefit. For our example we have a time line of five years, and it is divided into

- an annual premium fixed in non-indexed Swiss francs (CHF). We let the premium be constant for all years such that the premium $\Pi_t = \Pi$. As units for the premium we will use the zero-coupon bonds $Z^{(t)}$.
- survival benefit measured in an indexed fund $\mathbf{I}$ with price process $(I_t)_t$. This means that in this example the beneficiary does not receive a minimal guarantee in case of survival.
- death benefit defined as the indexed maximum of I_{t_i} and $(1+i)^{t_i-t_0}$, for $i \in (1, \ldots, 5)$ for a fixed minimum interest rate guarantee, i. We recognize this as a put option with strike $(1+i)^{t_i-t_0}$ at strike time t_i with value $\max(\mathbf{I}, (1+i)^{t_i-t_0})$ for $i \in (1, \ldots, 5)$. We denote the price of the put options $\mathrm{Put}_{t_i}(\mathbf{I}, (1+i)^{t_i-t_0})$.

We define our valuation vector with the following units

$$(\mathcal{U}_1, \ldots, \mathcal{U}_{11}) = (Z^{t_1}, \ldots, Z^{t_4}, \mathbf{I}, \mathrm{Put}(\mathbf{I}, (1+i)^{t_1-t_0}), \ldots, \mathrm{Put}(\mathbf{I}, (1+i)^{t_5-t_0})) \tag{7.1}$$

We will use this vector of units, and link them to financial instruments. This will enable us to calculate the market value of our valuation portfolio.

We end up with this valuation scheme:

time	premium	death benefit	survival benefit
t_0	$-l_{t_0} \cdot \Pi \cdot Z^{(t_0)}$		
t_1	$-l_{t_1} \cdot \Pi \cdot Z^{(t_1)}$	$d_{t_0} \cdot \big(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_1 - t_0})\big)$	
t_2	$-l_{t_2} \cdot \Pi \cdot Z^{(t_2)}$	$d_{t_1} \cdot \big(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_2 - t_0})\big)$	
t_3	$-l_{t_3} \cdot \Pi \cdot Z^{(t_3)}$	$d_{t_2} \cdot \big(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_3 - t_0})\big)$	
t_4	$-l_{t_4} \cdot \Pi \cdot Z^{(t_4)}$	$d_{t_3} \cdot \big(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_4 - t_0})\big)$	
t_5		$d_{t_4} \cdot \big(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_5 - t_0})\big)$	$l_{t_5} \cdot \mathbf{I}$

Table 7.1: Valuation scheme

7.2.2 Step 2

Now that we have found the units of our valuation portfolio we must find the number of each unit needed. From the table 7.1 above it is easy to find this number.

- We see that for the units $Z^{(t_0)}, \ldots, Z^{(t_4)}$, we need respectively $-l_{t_0} \cdot \Pi, \ldots, -l_{t_4} \cdot \Pi$ number of units.
- For the unit $\mathbf{I}$ we need l_{t_0} number of units since $l_{t_0} = d_{t_0} + d_{t_1} + d_{t_2} + d_{t_3} + d_{t_4} + l_{t_5}$ using the definition for l_x and d_x.
- For the put option price units $\mathrm{Put}(\mathbf{I}, (1+i)^{t_1 - t_0}), \ldots, \mathrm{Put}(\mathbf{I}, (1+i)^{t_5 - t_0})$ we need $d_{t_0}, \ldots, d_{t_4}$ number of units respectively.

Our cash flow is now mapped to the valuation portfolio, $\mathrm{VaPo}(\mathbf{X})$ which in turn maps the cash flow onto an insurance portfolio consisting of financial instruments. Here $\mathbf{X}$ is the cash flow of the insurance portfolio.

7.2.3 Step 3

We want a monetary value for the cash flow. This can be found by applying an accounting principal ε to the valuation portfolio, $\mathrm{VaPo}(\mathbf{X})$, to find the market value. As before the financial instruments will have to be valued based on its market price in the financial market. We obtain

$$\mathrm{Vapo}(\mathbf{X}) \mapsto \varepsilon(\mathrm{VaPo}(\mathbf{X})) \in \mathbb{R} \tag{7.2}$$

Now that the financial instruments are divided into units the understanding between actuaries and asset managers are eased. First the actuaries decompose the insurance portfolio into units. They now know how much is needed of each unit. Then the asset managers, or the same actuary, can evaluate these units using an accounting principle. For a cash flow $\mathbf{X}$, like the best estimate element we now have

$$Q(\mathbf{X}) = \varepsilon(\text{VaPo}(\mathbf{X})) = \sum_i \lambda_i(\mathbf{X}) \cdot \varepsilon(\mathcal{U}_i) \tag{7.3}$$

where λ_i represents the number of units $\mathcal{U}_i$.

7.2.4 Accounting principle

The accounting principle we use to comply with the solvency directive and regulatory requirements is the market value, ε. The units are zero coupon bonds, indexes, derivatives etc. The market value is in other words the price the asset can be exchanged for in a financial market. Depending on the time we want to value our portfolio we will use a time-dependent market value ε_s.

7.3 Stochastic life table

We call the best estimate of the liabilities in section 7.2 a deterministic model. In real life the life table is stochastic, and so are the liabilities. To add the stochastic risk to it, we therefore add a protection to the deterministic model. Examples of protection can be reinsurance products, risk loadings, or an extra capital buffer specially for technical risk.

So how much do we need in protection? We will need an expression that will involve the unknown mortality values. We let L_x denote a random number of individuals alive in year x, and D_x denote the random number of individuals who are alive in year x, and die before year (x+1). We then end up with a slightly different diagram than before. We use the same definition of the time, $t = [t_0, \ldots, t_5]$. Now we have a whole lot of unknowns. If we use d_{t_0} as predictor for D_{t_0} and use the fact that $d_{t_0} = l_{t_0} - l_{t_1}$ we obtain the prediction uncertainty for the mortality $D_{t_0} - d_{t_0} = l_{t_1} - L_{t_1}$.

If we define $p_x = \frac{l_{x+1}}{l_x}$ and $q_x = 1 - p_x$ we get the expression for the additional reserves we need. For t_1 we need an additional

$$(D_{t_0} - d_{t_0}) \cdot \left(\text{Put}(\mathbf{I}, (1+i)^{t_1 - t_0}) + \Pi \cdot Z^{(t_1)} - \frac{\text{Vapo}(\mathbf{X}_{(t_2)})}{l_{t_1}} \right) \tag{7.4}$$

in reserves where $\mathbf{X}_{(t_2)}$ refers to the cash flow $(0, 0, X_{(t_2)}, X_{(t_3)}, X(t_4), X_{(t_5)})$. If our stochastic mortality rate is higher than our deterministic rate several things occur. We need to pay more in death benefits at the specific time period. But due to the higher mortality we will have fewer contracts in our portfolio. Even though we receive less premium caused by the terminated contracts, higher mortality reduces our liabilities.

We therefore need to calculate with a new VaPo from time t_2. For $t_2, \ldots, t_5$ we need respectively

$$(D_{t_1} - q_{t_1} \cdot L_{t_1}) \cdot \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_2-t_0}) + \Pi \cdot Z^{(t_2)} - \frac{\mathrm{Vapo}(\mathbf{X}_{(t_3)})}{l_{t_2}}\right) \tag{7.5}$$

$$(D_{t_2} - q_{t_2} \cdot L_{t_2}) \cdot \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_3-t_0}) + \Pi \cdot Z^{(t_3)} - \frac{\mathrm{Vapo}(\mathbf{X}_{(t_4)})}{l_{t_3}}\right) \tag{7.6}$$

$$(D_{t_3} - q_{t_3} \cdot L_{t_3}) \cdot \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_4-t_0}) + \Pi \cdot Z^{(t_4)} - \frac{\mathrm{Vapo}(\mathbf{X}_{(t_5)})}{l_{t_4}}\right) \tag{7.7}$$

$$(D_{t_4} - q_{t_4} \cdot L_{t_4}) \cdot (\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_5-t_0}) - \mathbf{I} \tag{7.8}$$

in additional reserves. The added protection mentioned earlier can be given by a reinsurance premium loading like

$$\mathrm{RPP}_{t_0} = l_{t_0} \cdot (q^*_{t_0} - q_{t_0}) \cdot \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_1-t_0}) + \Pi \cdot Z^{(t_1)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(\mathrm{t}_2)})}{l_{t_1}}\right) \tag{7.9}$$

$$\mathrm{RPP}_{t_1} = l_{t_1} \cdot (q^*_{t_1} - q_{t_1}) \cdot \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_2-t_0}) + \Pi \cdot Z^{(t_2)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(\mathrm{t}_3)})}{l_{t_2}}\right) \tag{7.10}$$

$$\mathrm{RPP}_{t_2} = l_{t_2} \cdot (q^*_{t_2} - q_{t_2}) \cdot \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_3-t_0}) + \Pi \cdot Z^{(t_3)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(\mathrm{t}_4)})}{l_{t_3}}\right) \tag{7.11}$$

$$\mathrm{RPP}_{l_3} = l_{t_3} \cdot (q^*_{t_3} - q_{t_3}) \cdot \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_4-t_0}) + \Pi \cdot Z^{(t_4)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(\mathrm{t}_5)})}{l_{t_4}}\right) \tag{7.12}$$

$$\mathrm{RPP}_{t_4} = l_{t_4} \cdot (q^*_{t_4} - q_{t_4}) \cdot \big(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{t_5-t_0}) - \mathbf{I}\big) \tag{7.13}$$

where $q^*_x - q_x$ denotes the rate charged by the reinsurer subtracted the technical risk which combined equals the loading. Our valuation portfolio for stochastic mortality rate $\mathrm{VaPo}^{\mathrm{Prot}}$ can now be described as

$$\mathrm{VaPo}^{\mathrm{Prot}} = \mathrm{VaPo}(\mathbf{X}) + \sum_{t=t_0}^{t_4} \mathrm{RPP}_t \tag{7.14}$$

7.3.1 Accounting principle

As before the cash flow is decomposed into a vector whose basis consists of financial instruments. To find the monetary value of the reinsurance premium we can again apply the market value accounting principle, ε, to our new protected portfolio, the $\mathrm{VaPo}^{\mathrm{Prot}}$. The reinsurance premium valuated at time s is given by the equation (7.15) for a general x,

$$\begin{aligned}\Pi^{Re}_x &= \varepsilon_s(\mathrm{RPP}_x) \\ &= l_x \cdot (q^*_x - q_x) \\ &\cdot \varepsilon_s \left(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^{x-t_0+1}) + \Pi \cdot Z^{(x+1)} - \frac{\mathrm{VaPo}(\mathbf{X}_{(x+2)})}{l_{x+1}}\right)\end{aligned} \tag{7.15}$$

The market value of the valuation portfolio protected against technical risk at time t can be expressed by

$$\varepsilon_t(\mathrm{VaPo}^{\mathrm{Prot}}(\mathbf{X})) = \varepsilon_t(\mathrm{VaPo}(\mathbf{X})) + \sum_{x=t}^{t_4} \Pi_x^{Re} \tag{7.16}$$

By applying this accounting principal to our valuation portfolio onto the financial instruments, earlier defined as $\mathcal{U}_t$, it is possible to calculate a monetary value with the "total balance sheet" approach.

7.4 Conclusion

We have introduced a method of valuing our insurance portfolio with the help of financial instruments. There are however several simplifications in our method. The time period for the portfolio is predetermined. The modeling of options and guarantees might in fact become very difficult, e.g. incomplete markets. The choice of financial instruments to which we link the insurer's cash flow is arbitrary and we might as well choose other financial instruments.

However, the method illustrates how to divide the valuation portfolio into units of financial instruments. It creates a way for us to find the market value of our liabilities, which was our goal. This happens when we apply an accounting principal, and we obtain the monetary value of the valuation portfolio. To calculate the valuation portfolio protected against technical risk, we added the valuation portfolio and an additional reserve with a loading charged from the reinsurer. We have not specified how to find this loading, and it is yet another simplification in our method. In the next chapter we will look at a numerical example of the theory described. We will see how to use the steps given in this chapter, and determine the way the financial instruments and units are chosen.

8

NUMERICAL EXAMPLE OF A VALUATION PORTFOLIO

We are now ready to look at a numerical example of how to compute the valuation portfolio using realistic values. We follow the steps from chapter 7. We start out by assuming that we are dealing with a deterministic life table. Afterwards we will add a protection to our valuation portfolio to protect against technical risk in order to obtain the $\text{VaPo}^{\text{Prot}}$.

For our example we will consider a policy starting in the year 2000 and lasting until 2005. We will use the deterministic life table from table 8.1 of 1,000 policyholders in the starting year. We shall further assume a guaranteed interest rate of i = 3.00%, which was the guaranteed interest rate in 2000. The benefits follow an indexed fund **I** with $\text{I}_{2000} = 1$.

Time	Year	Survived l_{year}	Dead d_{year}
t_0	2000	$l_{t_0} = 1000$	
t_1	2001	$l_{t_1} = 996$	$d_{t_1} = 4$
t_2	2002	$l_{t_2} = 991$	$d_{t_2} = 5$
t_3	2003	$l_{t_3} = 986$	$d_{t_3} = 5$
t_4	2004	$l_{t_4} = 981$	$d_{t_4} = 5$
t_5	2005	$l_{t_5} = 975$	$d_{t_5} = 6$

Table 8.1: Example of a deterministic life table

Let us define a time index s which satisfies $t_0 \leq s < t_5$. We choose an equity-linked insurance contract with I_s denoting the price process of the equity index **I** valued at time s. We use the market value as our accounting principal, and let $\varepsilon_s(\mathbf{I})$ define the market value of **I** at time s.

We denote the price process of a zero coupon bond as $\mathbf{Z}^{(t)}$ with maturity in year t. We find its market value in year s by the following calculation

$$Z_s^{(t)} = Q(\mathbf{Z}^{(t)}|\mathcal{F}_s) = \varepsilon_s(Z^{(t)}) \text{ for } s < t \tag{8.1}$$

The market value of the zero coupon bond is given in terms of yield curves R(s,t) such that

$$Z_s^{(t)} = e^{-(t-s)\cdot R(s,t)} \tag{8.2}$$

The equity index and the yield curves R(s,t) are given in table 8.2 for years 2000-2005.

		R(s,t)				
Year, s	I_s	t-s = 1	t-s = 2	t-s = 3	t-s = 4	t-s = 5
2000	1	3.37%	3.52%	3.53%	3.56%	3.60%
2001	1.090	2.00%	2.85%	2.90%	2.96%	3.02%
2002	0.963	0.69%	1.84%	2.14%	2.38%	2.57%
2003	0.830	0.58%	0.79%	1.14%	1.46%	1.94%
2004	0.973	0.99%	1.11%	1.42%	1.70%	1.94%
2005	0.991	1.41%	1.14%	1.32%	1.48%	1.62%

Table 8.2: The equity index I_s, and yield curve R(s,t)

In order to find the an expression for the put option price we need to determine the dynamics of our equity index. We need to find the dynamics of our equity index, in order to find an expression for the put option price. The equity index is stochastic, with a random term. We start out by transforming the price process to

$$\tilde{I}_s = \frac{I_s}{Z_s^{(t)}}, \text{ for } s < t \text{ and fixed t} \tag{8.3}$$

By transforming the price process with a change of numeraire we are able to use our expression with a non-constant interest rate.

We shall assume $\tilde{I}_s$ is a martingale under the risk neutral measure and satisfies

$$d\tilde{I}_s = \sigma \tilde{I}_s dB_s, \tag{8.4}$$

where B_s is a Brownian motion under the risk neutral measure.

We want to find a general expression for the dynamics of $\tilde{I}_t$, which is the price process of the equity index at time t, valued at time s. In order to compute the dynamics of $\tilde{I}_t$ we will use Itô's lemma. We start out with the two equations

$$d\tilde{I}_t = \sigma \tilde{I}_t dB_t \tag{8.5}$$

$$\begin{aligned} d\tilde{I}_t^2 &= \sigma^2 \tilde{I}_t^2 dB_t^2 \\ &= \sigma^2 \tilde{I}_t^2 dt, \text{ since } dB_t^2 := dt \end{aligned} \tag{8.6}$$

and use Itô's lemma on the expression ($\ln \tilde{\mathrm{I}}_t$)

$$
\begin{aligned}
d(\ln \tilde{\mathrm{I}}_t) &= 0 + \frac{1}{\tilde{\mathrm{I}}_t} d\tilde{\mathrm{I}}_t - \frac{1}{2}\frac{1}{\tilde{\mathrm{I}}_t^2} d\tilde{\mathrm{I}}_t^2 \\
&= \frac{1}{\tilde{\mathrm{I}}_t} d\tilde{\mathrm{I}}_t - \frac{1}{2}\frac{\sigma^2 \tilde{\mathrm{I}}_t^2}{\tilde{\mathrm{I}}_t^2} dt \qquad (8.7) \\
&= \sigma dB_t - \frac{1}{2}\sigma^2 dt, \qquad (8.8)
\end{aligned}
$$

where we in equation (8.7) have inserted the equations (8.5) and (8.6). To determine the dynamics of the price process $\tilde{I}_t$, which starts at time s, we need to integrate equation (8.8).

$$
\begin{aligned}
\int_s^t d(ln\tilde{\mathrm{I}}_t) &= \int_s^t dB_u - \frac{1}{2}\int_s^t \sigma^2 du \\
ln(\tilde{\mathrm{I}}_t)_s^t &= \sigma(B_u)_s^t - \frac{1}{2}\sigma^2(t-s) \\
ln(\frac{\tilde{\mathrm{I}}_t}{\tilde{\mathrm{I}}_s}) &= \sigma B_{t-s} - \frac{1}{2}\sigma^2(t-s) \\
\tilde{\mathrm{I}}_t &= \tilde{\mathrm{I}}_s e^{-\frac{1}{2}\sigma^2(t-s)+\sigma B_{t-s}} \qquad (8.9)
\end{aligned}
$$

With the dynamics of $\tilde{\mathrm{I}}_t$, we are ready to find an expression for the put option price. We define $\epsilon \sim \mathcal{N}(0,1)$ which has the distribution

$$
q(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \qquad (8.10)
$$

Since σB_{t-s} has the distribution $\mathcal{N}(0, \sigma^2(t-s))$, σB_{t-s} can also be expressed by the equivalent distribution $\sigma\sqrt{t-s} \cdot \epsilon$, because of the definition of ϵ. The put option price formula is then given by

$$
\begin{aligned}
\mathrm{Put}_s(\mathbf{I}, (1+i)^{t-t_0}) &= Z_s^{(t)} E_{P^*}\left[\max\left((1+i)^{t-t_0} - \tilde{I}_t\right) |\mathcal{F}_s\right] \\
&= Z_s^{(t)} E_{P^*}\left[\max\left((1+i)^{t-t_0} - \tilde{I}_s e^{-\frac{\sigma^2}{2}(t-s)+\sigma\sqrt{t-s}\cdot\epsilon}\right) |\mathcal{F}_s\right] \\
&= Z_s^{(t)} \int_{\{(1+i)^{t-t_0} > \tilde{I}_t\}} \left((1+i)^{t-t_0} - \tilde{I}_s e^{-\frac{\sigma^2}{2}(t-s)+\sigma\sqrt{t-s}\cdot y}\right) q(y)dy
\end{aligned}
\qquad (8.11)
$$

where we valuate the expression at time s.

By solving the integration limit $\{(1+i)^{t-t_0} > \tilde{I}_s e^{-\frac{1}{2}\sigma^2(t-s)+\sigma\sqrt{t-s}\cdot y}\}$ with respect to y, we find the upper limit, a, of the integration. The upper limit is given by

$$
a = -\frac{\ln\left(\frac{\tilde{\mathrm{I}}_s}{(1+i)^{t-t_0}}\right) + \frac{1}{2}\sigma^2(t-s)}{\sigma\sqrt{t-s}} > y \qquad (8.12)
$$

We continue by dividing equation (8.11) into two expressions with the new limits. First

$$Z_s^{(t)} \int_{-\infty}^{a} (1+i)^{t-t_0} q(y) dy = Z_s^{(t)} (1+i)^{t-t_0} \cdot \Phi(a), \tag{8.13}$$

where $\Phi(\cdot)$ is the cumulative distribution function for a standard normal distribution such that $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{y^2}{2}} dy$. The second expression is a little more complicated

$$\begin{aligned}
&Z_s^{(t)} \int_{-\infty}^{a} \tilde{\mathrm{I}}_s e^{-\frac{\sigma^2}{2}(t-s)+\sigma\sqrt{t-s}\cdot y} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= Z_s^{(t)} \cdot \tilde{\mathrm{I}}_s \int_{-\infty}^{a-\sigma\sqrt{t-s}} \frac{1}{\sqrt{2\pi}} e^{\frac{v^2}{2}} dv, \text{ where } v = y - \sigma\sqrt{t-s} \\
&= Z_s^{(t)} \cdot \tilde{\mathrm{I}}_s \cdot \Phi(-\sigma\sqrt{t-s} + a) \qquad (8.14)
\end{aligned}$$

Using equation (8.3) we get the following expression for the put option price

$$\begin{aligned}
\mathrm{Put}_s(\mathbf{I}, (1+i)^{t-t_0}) = Z_s^{(t)} \cdot (1+i)^{t-t_0} \cdot \Phi\left(\frac{-\ln\left(\frac{\mathrm{I}_s}{(1+i)^{t-t_0} Z_s^{(t)}}\right) + \frac{\sigma^2}{2}(t-s)}{\sigma\sqrt{t-s}} \right) \\
- \mathrm{I}_s \cdot \Phi\left(\frac{-\ln\left(\frac{\mathrm{I}_s}{(1+i)^{t-t_0} Z_s^{(t)}}\right) - \frac{\sigma^2}{2}(t-s)}{\sigma\sqrt{t-s}} \right) \qquad (8.15)
\end{aligned}$$

We will use a guaranteed interest rate of i = 3%. Index **I** is described in table 8.2 and $\sigma = 0.15$ was used. Hence we have the following put option prices valued from years 2000 through 2004, calculated for the remaining years of the policy for all years.

s	t				
	2001	2002	2003	2004	2005
2000	0.053	0.069	0.080	0.088	0.093
2001		0.034	0.051	0.066	0.076
2002			0.117	0.131	0.144
2003				0.249	0.267
2004					0.140

Table 8.3: Put option prices valued at time s, with strike time t

Now we can calculate the premium Π for the valuation portfolio $\mathbf{X}$. The premium and the expenses paid must be equal seen from time t_0 by the premium equivalence principle.

Hence

$$\begin{aligned}
\varepsilon_{t_0}(\mathrm{VaPo}(\mathbf{X})) = \varepsilon_{t_0}(&-(l_{t_0}Z^{t_1} + l_{t_1}Z^{t_2} + l_{t_2}Z^{t_3} + l_{t_3}Z^{t_4} + l_{t_4}Z^{t_4}) \cdot \Pi \\
&+ d_{t_1}(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i))) + d_{t_2}(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^2)) \\
&+ d_{t_3}(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^3)) + d_{t_4}(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^4)) + \\
&+ d_{t_5}(\mathbf{I} + \mathrm{Put}(\mathbf{I}, (1+i)^5)) + l_{t_5} \cdot \mathbf{I}) \qquad (8.16) \\
&= 0 \qquad (8.17)
\end{aligned}$$

We solve for Π given that

- both the survival and death capital at time t_0 was 100,000 of an arbitrary currency.
- the market value of the index $\mathbf{I}$ is given by $\varepsilon_{t_0}(\mathbf{I}) = \mathrm{I}_{t_0}$.
- the market value of the zero coupon bond Z^t is given by $\varepsilon_{t_0}(Z^t) = Z^t_{t_0}$
- the market value of the put option is given by $\varepsilon_{t_0}(\mathrm{Put}(\mathbf{I}, (1+i))) = \mathrm{Put}_{t_0}(\mathrm{I}_{t_0}, (1+i))$.

We obtain a premium of $\Pi = 21,675$. Now we have all the results we need to be able to calculate the valuation portfolio cash flow valuated for each year. For year s the value of the valuation portfolio is given by

$$\varepsilon_s(\mathrm{VaPo}(\mathbf{X}_{(s+1)}) - \Pi \cdot l_s \cdot Z^{(s)})) \qquad (8.18)$$

where $\mathbf{X}_{(s+1)}$ is all of the remaining cash flows from time s+1. Mathematically, $\mathbf{X}_{(s+1)} = (0, \ldots, X_{s+1}, \ldots X_t)$. If we let the time run between $s = (2000, \ldots, 2004)$ then the value of the valuation portfolio for each year is given by

s	ε_s
2000	0
2001	26,376,842
2002	32,454,986
2003	39,656,224
2004	74,268,450

Table 8.4: Market value of valuation portfolio cash flow at time s

Taking into account to a stochastic mortality, we want to find the reinsurance premium charged by a reinsurer. We use equation (7.15), and the market value at time t_0, ε_{t_0}. For our example, we use the stress parameter from the Solvency II directive, an increase in the mortality rate of 10% for each year. This gives us $q^*_t = 1.1 \cdot q_t$.

s	Π_s^{Re}
2000	33,635
2001	32,778
2002	23,626
2003	14,854
2004	6,907

Table 8.5: Reinsurance premium

Our insurance portfolio only lasts for five year. We see that the closer we get to the end, the reinsurance premium charge decreases. The reason is that all of the risk factors decreases.

8.1 Conclusion

By following the steps and dividing the valuation portfolio into units of financial instruments, valuation becomes a pretty straight forward computation. The computation of the reinsurance premium can also be found without significant struggle. There are however simplifications in the calculation.

- The life table and mortality table will not be given by a discrete table, but rather as a survival probability formula depending on age for each individual. See discussions in chapter 5.
- We have assumed that the fixed minimal guarantee rate i is a constant figure. This might be true for shorter time periods such as 5-10 years, but will probably change over a longer period depending on the financial market. If the market is in trouble, the guarantee can be impossible to achieve. And an insurance company will have to pay higher and higher put option prices.
- In order to compensate for losses, both financial and technical, the insurance company can adjust the premium. I have assumed that the premium stays constant throughout the policy lifetime.
- Maybe the most crucial of the simplifications is that we have assumed that we can find all the financial instruments we need in the financial market. This is probably a too optimistic assumption.
- The reinsurance loading is defined as a constant factor times the mortality probability. In reality these loadings may be negotiated and changed each year depending on last years results. It is however realistic that the loading should be a more complex function of the mortality probability.

9

ASSET LIABILITY MANAGEMENT

In the previous chapter we have found expressions for two different valuation portfolios. The VaPo with deterministic mortality, and the $\text{VaPo}^{\text{Prot}}$ which is the portfolio with stochastic mortality protected against technical risks with an extra reinsurance loading. I will consider the latter in this chapter. This chapter deals with the asset and liability management of insurance portfolios based on the construction of "solvency strategies" as proposed by Bühlmann et al. [17]. We will assume that the financial instruments we need for the construction of the $\text{VaPo}^{\text{Prot}}$ can be bought in the assets market. We will compare the $\text{VaPo}^{\text{Prot}}$ to a real asset portfolio S of the insurance undertaking. S has to be greater or equal to the valuation portfolio to obtain solvency.

9.1 Market risk

Market risk is another expression for financial risk which is the risk of the actual investment return being less than the expected return. If we were to buy the exact units of financial instruments of which the $\text{VaPo}^{\text{Prot}}$ consists, we would have avoided all market risk. And since the $\text{VaPo}^{\text{Prot}}$ is protected against technical risk, there would be no risk. There are however reasons for not applying this strategy. For one, companies are interested in maximizing their financial return. Taking additional market risk opens the possibility for higher returns.

9.1.1 Solvency definition

Before I describe how the asset portfolio should look like we need to define solvency according to the Solvency II directive. We use the economic accounting principal ε_s.

As before ε_s corresponds to the market price at time s paid at the asset market for the financial instruments. At time t_0 a company is solvent if

$$\varepsilon_{t_0}(\mathrm{S}) \geq \varepsilon_{t_0}(\mathrm{VaPo}^{\mathrm{Prot}}) \tag{9.1}$$

But a company should be solvent through all the time periods ahead, so from an overall perspective of an insurance portfolio the following must hold

$$\varepsilon_s(\mathrm{S}) \geq \varepsilon_s(\mathrm{VaPo}^{\mathrm{Prot}}) \qquad \forall \quad s > t_0 \tag{9.2}$$

The Solvency II directive divide the time period into time intervals of one year when considering solvency. So for the accounting condition we make sure we are solvent for the time period $[t_0, t_0+1]$. Then at time (t_0+1) we iterate the procedure and find a new accounting condition. Therefore the Solvency II definition is decoupled into an iterative process. This iteration may cause problems in practice when trying to find market values for assets which have maturity many years ahead. It is however the definition we will use to comply with the solvency directive.

9.1.2 ALM mismatch

The big question is how to choose the asset portfolio S. One suggestion is to define S such that

$$S = \mathrm{VaPo}^{\mathrm{Prot}} + \mathrm{F} \tag{9.3}$$

where $\mathrm{VaPo}^{\mathrm{Prot}}$ is self-financing, and F is excess capital which has to be positive. From a mathematical point of view this would ensure solvency, and shows that solvency is possible. In real life however there are a few problems with this portfolio construction. One problem is that the financial instruments might not be available for purchase at all times even though they exist i.e. the put option. Another problem arises when we have very long term liabilities. It is often impossible to find financial instruments with such long time perspectives as 30 years ahead, especially in smaller countries and financial communities such as Norway. The fact remains that S should not actually contain $\mathrm{VaPo}^{\mathrm{Prot}}$. To maximize returns a company will take higher market risk causing a mismatch between the asset portfolio and their liabilities. This mismatch is called ALM mismatch.

If we cannot find the long term assets we need in order to replicate our liabilities we use a strategy that leads to mismatch. We build ourselves a replicating portfolio which consists of assets with shorter maturities than our liabilities, and rebalance the portfolio at maturity. A replicated portfolio means a portfolio that is possible to construct using instruments from the financial market.

Mismatch between the asset portfolio and the company's liabilities will cause higher market risk, and the company is, by the Solvency II directive, required to hold additional capital for protection against market risk. In life insurance protection against market risk turns out to be by far the most dominant requirement for capital, even compared to the capital required for protection against technical risk, see section 4.1.1, second paragraph.

9.2 Marbrage option

We want to use a financial instrument to ensure solvency. The goal is to be able to protect us against unfavorable events, while maximizing returns. We will again assume that the financial instruments included in the $\mathrm{VaPo}^{\mathrm{Prot}}$ are available in the assets market. Then there exists a strategy that leads to a valuation portfolio that protects the undertaking against market risk. This strategy is to define the asset portfolio S as

$$\mathrm{S} = \tilde{\mathrm{S}} + \mathrm{M} + \mathrm{F} \tag{9.4}$$

where $\tilde{S}$ is any asset portfolio which satisfies the solvency requirement, equation (9.2). M is the price needed to buy a Marbrage option, and F is excess capital which is required to be strictly positive at all times. A Margrabe option is a financial instrument that gives the buyer the right to exchange one asset for another. In our case we want the possibility of switching from $\tilde{\mathrm{S}}$ to $\mathrm{VaPo}^{\mathrm{Prot}}$ at the end of every accounting year. In order to do this we buy a Margrabe option with price M. In our case it gives us the right to switch from the asset portfolio $\tilde{\mathrm{S}}$ to the $\mathrm{VaPo}^{\mathrm{Prot}}$ whenever

$$\varepsilon_{t_0+1}(\mathrm{VaPo}^{\mathrm{Prot}}) > \varepsilon_{t_0+1}(\tilde{\mathrm{S}}) \tag{9.5}$$

The valuation portfolio protects both against technical risk and market risk and is with the use of a Margrabe option equal to $\tilde{\mathrm{S}} + M$. For valuation at time t this implies

$$\mathrm{Prot} = \varepsilon_t(\tilde{\mathrm{S}}) + M_t \tag{9.6}$$

The Margrabe option enables us to take risks with our asset portfolio, $\tilde{\mathrm{S}}$, by ALM mismatch. If our risky strategy fails we can switch assets. We are therefore able to cover our liabilities regardless of the outcome, hence solvency is guaranteed.

We now decouple the solvency problem into recursive one-period problems, where we each year spend the price of a Margrabe option in order to be covered. This covers the mismatch between the $\mathrm{VaPo}^{\mathrm{Prot}}$ and $\tilde{\mathrm{S}}$.

The Margrabe option price is derived by Gerber and Shiu [8] and is given by the formula (9.9) where

$$x_1 = \varepsilon_t[VaPo^{Prot}] \tag{9.7}$$

$$x_2 = \varepsilon_t[\tilde{S}] \tag{9.8}$$

the valuation occurs at time s for a price at time t, and

$$M_t = x_1\Phi(d_1) - x_2\Phi(d_2) \tag{9.9}$$

$$d_1 = \frac{\ln(\frac{x_1}{x_2}) + \frac{v^2}{2}(t-s)}{v\sqrt{t-s}} \tag{9.10}$$

$$d_2 = d_1 - v\sqrt{t-s} \tag{9.11}$$

Φ is as before the cumulative standard normal density function, and $v^2 = \sigma_1^2 - 2\sigma_1 v\sigma_2\rho + \sigma_2^2$ is the variance of $(x_1/x_2)^{-1}d(x_1/x_2)$.

To derive the Margrabe option price Gerber and Shiu have switched to an equivalent martingale measure in order to achieve a martingale option price, see details in Appendix C. The transformation they have used is called an Esscher transform. This transformation is a well-known method used in actuarial mathematics, and is a useful technique to obtain a reasonable fair value equivalent martingale measure. The Esscher transform allows us to find a deflator which achieves the martingale property. In a complete arbitrage-free market there exists only one unique equivalent martingale measure. If we on the other hand operate in an incomplete financial market, the equivalent martingale measure is no longer unique. In this case there exist several equivalent martingale measures such as a minimal entropy martingale measure or Esscher martingale measure. The minimal entropy martingale measure is preferred because of the large deviation theory. According to Miyahara [12] and Sanov's theorem

> the minimal entropy martingale measure is the most possible empirical probability measure of paths of price process in the class of the equivalent martingale measures.

The theory of the minimal entropy measure is beyond the scope of this thesis. The Esscher transform is often not applicable to heavy-tailed distributions. Therefore this calculation is not suitable for the non-life insurance industry, but in life insurance its use can be justified.

9.3 Hedging Margrabe option

It may often not be possible to actually buy the Margrabe option for all time periods. The option might not even exist in an incomplete market. Therefore we can construct a strategy where we end up with the value of the Prot from equation (9.6). This strategy is called hedging. Let us define a function H

$$H(t,x) = x \cdot \Phi\left(\frac{\log x + \frac{v^2}{2}(t-s)}{v\sqrt{t-s}}\right) - \Phi\left(\frac{\log x - \frac{v^2}{2}(t-s))}{v\sqrt{t-s}}\right) \tag{9.12}$$

where x follows a Brownian motion. Then we have that

$$M_t = H\left(t, \frac{x_1}{x_2}\right) \cdot x_2 \tag{9.13}$$

where x_1 and x_2 are defined in (9.7) and (9.8), and M_t is defined in (9.9). To find a hedging strategy HS(t,x), we use Itô's lemma

$$\partial HS(t,x) = \frac{\partial H(t,x)}{\partial t}dt + \frac{\partial H(t,x)}{\partial x}dx + \frac{1}{2}\frac{\partial^2 H(t,x)}{\partial x^2}(dx)^2 \tag{9.14}$$

Since x follows a Brownian motion $(dx)^2 = dt$. We want the expression to be a martingale under P. For that to be the case, the dt terms must equal zero. We therefore want to use $\frac{\partial H(t,x)}{dx}$ in our hedging strategy. This expression equals

$$\begin{aligned}
\frac{\partial H(t,x)}{\partial x} &= \Phi\left(\frac{\log x + \frac{v^2}{2}(t-s)}{v\sqrt{t-s}}\right) + \phi\left(\frac{\log x + \frac{v^2}{2}(t-s)}{v\sqrt{t-s}}\right)/(v\sqrt{t-s}) \\
&- \phi\left(\frac{\log x - \frac{v^2}{2}(t-s)}{v\sqrt{t-s}}\right)/(x \cdot v\sqrt{t-s}) \qquad (9.15) \\
&= \Phi\left(\frac{\log x + \frac{v^2}{2}(t-s)}{v\sqrt{t-s}}\right) \qquad (9.16)
\end{aligned}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ is the derivative function of $\Phi(x)$. We decide to invest

$$a = \Phi\left(\frac{\log(\frac{x_1}{x_2}) + \frac{v^2}{2}(t-s)}{v\sqrt{t-s}}\right) \tag{9.17}$$

in $x_1 = \varepsilon_t[VaPo^{Prot}]$ and

$$b = 1 - \Phi\left(\frac{\log(\frac{x_1}{x_2}) - \frac{v^2}{2}(t-s)}{v\sqrt{t-s}}\right) \tag{9.18}$$

in $x_2 = \varepsilon_t[\tilde{S}]$. We recognize $a = \Phi(d_1)$, and $b = 1 - \Phi(d_2)$. We get

$$\begin{aligned}
a \cdot x_1 + b \cdot x_2 = x_2 + M_t &= \varepsilon_t(\tilde{\mathrm{S}}) + M_t \\
&= \mathrm{Prot} \qquad (9.19)
\end{aligned}$$

From equation (9.6). We have found a hedging strategy which leaves us with the exact same amount as if were to buy the Margrabe option. Hence solvency is ensured.

9.4 Risk bearing capital

Another possibility instead of buying the Margrabe option is to finance a risk bearing capital. We can e.g. use expected shortfall or the value at risk as our risk measure. For a given VaR-measure ϵ we choose q_{t_0} such that have enough capital to satisfy

$$P^* \left[(1 + q_{t_0}) \frac{\varepsilon_{t_0}(\tilde{S})}{\varepsilon_{t_0}(\text{VaPo}^{\text{Prot}})} \geq 1 | \mathcal{F}_{t_0} \right] \geq 1 - \epsilon, \tag{9.20}$$

where P^* is the equivalent martingale measure. The risk bearing capital approach is not optimal because we do not have the possibility of switching assets if the results one year is catastrophic. The probability of having too little capital is, on the other hand, very small depending on which ϵ we choose. A natural choice could be $\epsilon = 0.01$. This method of covering risk is a practical approach, and is applied under the Swiss Solvency Test. We find the expression for q_{t_0} by switching the inequality

$$1 - P \left[\log \left(\frac{\varepsilon_{t_0}(\tilde{S})}{\varepsilon_{t_0}(\text{VaPo}^{\text{Prot}})} \right) \leq -\log(1 + q_{t_0}) | \mathcal{F}_{t_0} \right] = 1 - \epsilon \tag{9.21}$$

then by subtracting with the expected investment rate and dividing by the standard deviation we get a normalization

$$\Phi \left(\frac{-\log(1 + q_{t_0}) - \mu}{\sigma} \right) = \epsilon \tag{9.22}$$

and we finally achieve the expression

$$q_{t_0} = 1 - \exp(-\sigma \cdot \Phi^{-1}(\epsilon) + \mu) \tag{9.23}$$

We have found an expression with two unknowns, both the expectation of the investment return, and the standard deviation. We can set the expectation equal the risk free rate. The standard deviation should be a function of the expectation since higher return increase the volatility.

9.5 Conclusion

Our main goal is to remain solvent. But we also want to maximize our investment returns. We have looked at three methods of achieving this, either by buying a Margrabe option, by hedging a Margrabe option or by setting aside enough risk bearing capital.

The first two choices take use of the financial market, and is therefore automatically market consistent. The problem with buying the Margrabe option is that it might not exist in the financial market. Therefore we looked at the option of hedging the option.

These two methods solve our solvency problem by each year spending the price of the Margrabe option to protect against financial risk. The price found measures the ALM mismatch between the real asset portfolio, and the liability portfolio. The last choice takes use of a risk measure and uses a quantile hedging approach. This method is not favorable because it is not market consistent, and is not dependent on the financial market.

Appendix A

LIBOR forward rate

A.1 Evolution of the LIBOR forward rate

The method is proposed by Proske [15]. The equation (6.17) can be written as

$$L(t,T) = L(0,T) + \int_0^t \gamma(s,T) \cdot L(s,T) d\tilde{B}_s^{T+\delta} \tag{A.1}$$

This is the expression we want to find. We apply Itô's formula on (A.2), as described in section 6.6.3. We set

$$Y_t = g(t,x) = L(0,T_j) \cdot e^x \tag{A.2}$$

$$X_t = \int_0^t \gamma(s,T_j) d\tilde{B}_s^{T_{j+1}} - \frac{1}{2}\int_0^t \gamma^2(s,T_j) ds \tag{A.3}$$

Then we see that

$$dX = \gamma(s,T_j) d\tilde{B}_s^{T_{j+1}} - \frac{1}{2}\gamma^2(s,T_j) ds \tag{A.4}$$

$$(dX)^2 = \gamma^2(s,T_j) ds \tag{A.5}$$

Itô's formula on Y_t has the solution

$$Y_t = L(0,T_j) + \int_0^t \frac{\partial Y_s}{\partial s} ds + \int_0^t \frac{\partial Y_s}{\partial X} dX + \frac{1}{2}\int_0^t \frac{\partial^2 Y_s}{\partial X^2}(dX)^2 \tag{A.6}$$

$$= L(0,T_j) + \int_0^t Y_s \gamma(s,T_j) d\tilde{B}_s^{T_{j+1}} \tag{A.7}$$

where

$$Y_s = L(s,T) = L(0,T_j) e^{X_t} \tag{A.8}$$

We have now derived the expression (A.1) we wanted to show.

A.2 Simulation formula

We take in use equation (6.14) and state that

$$d\tilde{B}_t^{T_j} := d\tilde{B}_t + \sigma^*(t, T_j)dt \tag{A.9}$$

$$d\tilde{B}_t^{T_{j+1}} := d\tilde{B}_t + \sigma^*(t, T_{j+1})dt \tag{A.10}$$

We subtract the equations

$$d\tilde{B}_{T_j} = (\sigma^*(t, T_j) - \sigma^*(t, T_{j+1}))dt + \tilde{B}_t^{T_{j+1}} \tag{A.11}$$

We insert the expression found for gamma, (6.16)

$$d\tilde{B}_t^j = \left(-\frac{\delta\gamma(t, T_j)L(t, T_j)}{1 + \delta L(t, T_j)}\right) dt + d\tilde{B}_t^{T_{j+1}} \tag{A.12}$$

$$d\tilde{B}_t^n = \left(-\frac{\delta\gamma(t, T_n)L(t, T_n)}{1 + \delta L(t, T_n)}\right) dt + d\tilde{B}_t^{T_{n+1}} \tag{A.13}$$

$$d\tilde{B}_t^{n-1} = \left(-\frac{\delta\gamma(t, T_{n-1})L(t, T_{n-1})}{1 + \delta L(t, T_{n-1})}\right) dt + d\tilde{B}_t^{T_n} \tag{A.14}$$

$$\vdots \tag{A.15}$$

$$d\tilde{B}_t^{j+1} = -\sum_{i=j+1}^{n} \frac{\delta\gamma(t, T_i)L(t, T_i)}{1 + \delta L(t, T_i)} dt + d\tilde{B}_t T_{n-1} \tag{A.16}$$

for $j+1 \leq n$. We use $\tilde{\gamma}(t, T_j) \approx \gamma(t, T_j)$ and A.8 and insert $\tilde{B}_s^{T_{j+1}}$ into X_t from equation (A.3) and end up with

$$\begin{aligned} L(t, T_j) = L(0, T_j)exp\{&\int_0^t \tilde{\gamma}(s, T_j)d\tilde{B}_s^{T_{n+1}} \\ &+ \int_0^t (\tilde{\gamma}(s, T_j)(-\textstyle\sum_{i=j+1}^{n} \displaystyle\frac{\delta\tilde{\gamma}(s, T_i)}{1 + \delta L(s, T_i)}) - \frac{1}{2}(\tilde{\gamma}(s, T_j))^2)ds\} \end{aligned} \tag{A.17}$$

which is the simulation formula for the LIBOR forward rate.

Appendix B

LIBOR forward rate plots

The tenor used is $\delta = \frac{1}{4}$. The three time periods considered are T_1, T_2 and T_3. The simulations are divided into three combined steps. The volatility parameters given are

$$\tilde{\gamma}(t, T_1) = 16.05\%, \quad \tilde{\gamma}(t, T_2) = 16.12\%, \quad \tilde{\gamma}(t, T3) = 16.19\% \tag{B.1}$$

The data is collected from Bloomberg in June 2009. The underlying Brownian motion used was $\tilde{B}_t^{T_4}$. The superscript T_4 demonstrates that the same Brownian motion is used for all of the three time periods. The Brownian motion is actually simulated one whole year ahead. $\tilde{B}_t^{T_4}$was found by simulating

$$B_i = B_{i-1} + \epsilon \cdot \sqrt{1/N} \tag{B.2}$$

where $B_0 = 0$, N where the amount of points chosen, and ϵ ∼N(0,1). The LIBOR is measured in US dollars. The LIBOR interest rates where $L(0, T_3) = 1.85\%, L(0, T_2) = 1.74\%$ og $L(0, T_1) = 1.50\%$ respectively. I have simulated 20 possible paths for the forward LIBOR rate in each of the three steps.

The exercise shows the possibility of simulating LIBOR forward rates.

B.1 Step 1

The first step corresponds to the period $0 \leq t \leq T_3$ which is the time period up to 9 months.

$$L(t, T_3) = L(0, T_3) \cdot \exp\left\{16.19\% \cdot \tilde{B}_t^{T_4} - \frac{1}{2} \cdot t \cdot (16.19\%)^2\right\} \tag{B.3}$$

I have used 750 points to simulate a brownian motion.

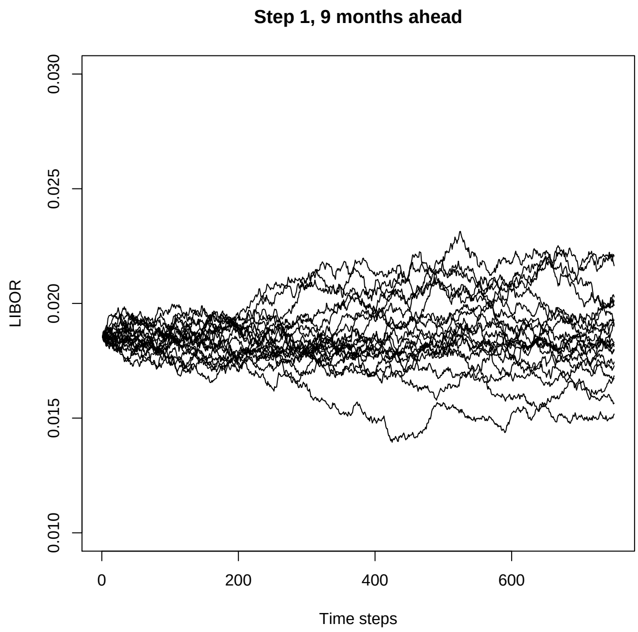

Figure B.1: 9 months LIBOR rate

B.2 Step 2

The second step corresponds to the period $0 \leq t \leq T_2$ which is the time period up to 6 months. The simulation formula is

$$L(t,T_2) = L(0,T_2)\exp\left\{16.12\% \cdot \tilde{B}_t^{T_4} + \int_0^t \left(16.12\% \cdot -(\frac{\delta \cdot 16.9\% \cdot L(s,T_3)}{1+\delta \cdot L(s,T_3)})\right) ds \right.$$
$$\left. - \frac{1}{2} \cdot t \cdot (16.12\%)^2 \right\}$$

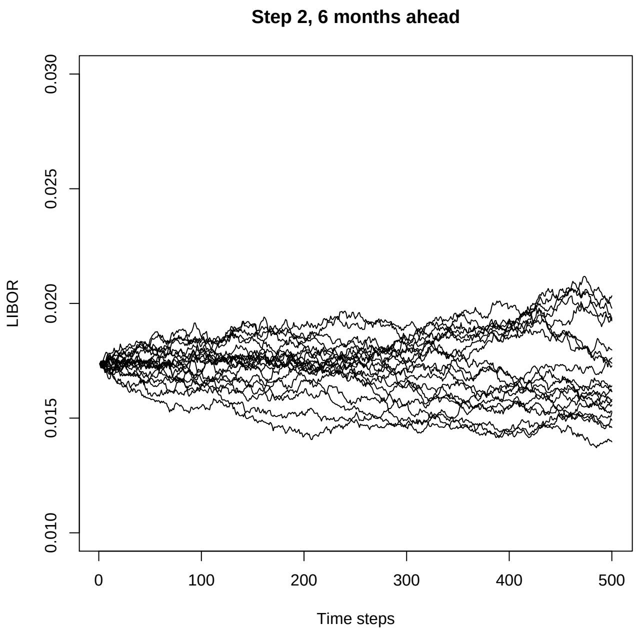

Figure B.2: 6 months LIBOR rate

B.3 Step 3

The last step corresponds to the period $0 \leq t \leq T_1$ which is the time period up to 3 months. The formula is given by

$$L(t,T_1) = L(0,T_1)\exp\Bigg\{16.05\% \cdot \tilde{B}_t^{T_4} + \int_0^t \left(16.05\% \cdot (-\frac{\delta \cdot 16.12\% \cdot L(s,T_2)}{1+\delta L(s,T_2)} - \frac{\delta \cdot 16.19\% \cdot L(s,T_3)}{1+\delta L(s,T_3)})\right) ds - \frac{1}{2} \cdot t \cdot (16.05\%)^2\Bigg\}$$

The variation between the simulations become smaller with shorter time periods, as should be expected.

B.4 Against each other

Finally I plotted one simulation for each of the three time periods together in a single plot. They follow each other closely due to the same underlying brownian motion, even though they have different starting values, and expressions.

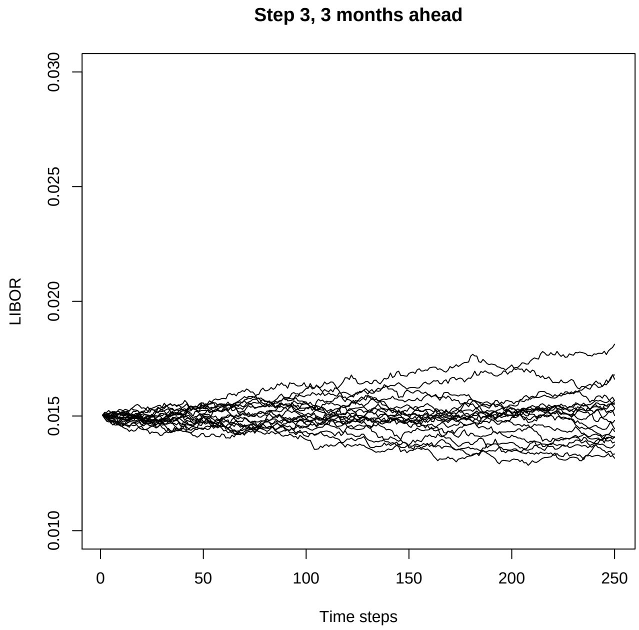

Figure B.3: 3 months LIBOR rate

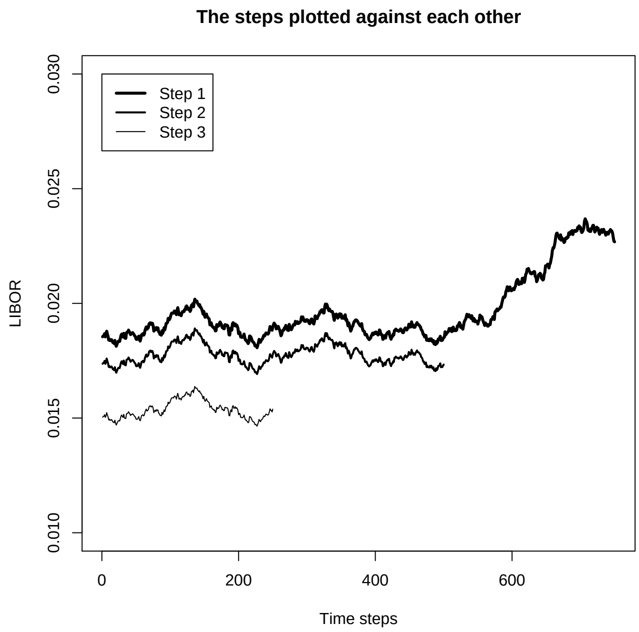

Figure B.4: The three time period in the same plot

Appendix C

Esscher transform

The Esscher transform is an actuarial technique, and is a powerful valuation tool (in incomplete markets). We will see how it helps us find an equivalent martingale measure to a discounted price process.

Let us consider a general stochastic process x(t), $t > 0$, with stationary and independent increments, and initial value x(0) = 0. Its price is growing with an interest rate which follows a stochastic process denoted by Y(t).

$$x(t) = x(0) \cdot e^{Y(t)} \tag{C.1}$$

We assume that the random variable Y(t) has a probability density function f(y,t). The moment generating function M(z,t) of Y(t) is defined by

$$M(z,t) = E[e^{(zY(t))}] = \int_{-\infty}^{\infty} e^{zy} f(y,t)dy \tag{C.2}$$

The Esscher transform with parameter h of f is given by

$$f(y,t;h) = \frac{e^{hy} f(y,t)}{M(h,t)} \tag{C.3}$$

The corresponding moment generating function of $f(y,t;h)$ is

$$M(z,t;h) = \int_{-\infty}^{\infty} e^{zy} f(y,t;h)dy \tag{C.4}$$

$$= \frac{M(z+h,t)}{M(h,t)} \tag{C.5}$$

Since M(z,t) is continous in t, it can be shown that

$$M(z,t;h) = [M(z,t;h)]^t \tag{C.6}$$

In order to find the risk neutral Esscher transform we want to find a $h = h^*$ such that the discounted price process $e^{\delta t}x(t)$ is a martingale with respect to the probability measure corresponding to h^*. The probability density function for E^* is $f(y, t; h^*)$. δ denotes the risk-free force of interest. We want

$$x(0) = E^*[e^{\delta t}x(t)] = e^{\delta t}E^*[x(t)] \tag{C.7}$$

Using equation (C.1), h^* is the solution of

$$1 = e^{\delta t}E^*[e^{Y(t)}] \quad \text{, or} \tag{C.8}$$

$$e^{\delta t} = M(1, t; h^*) \tag{C.9}$$

Setting t = 1, we obtain,

$$\delta = ln[M(1, 1; h^*)] \tag{C.10}$$

It can be shown that there is a unique solution $h^* \in \mathbb{R}^L$ which satisfies (C.9) for all t $=1, \ldots, L$. Hence we have found the discount factor δ which gives an equivalent martingale measure for the price process.

Appendix D

R-code

D.1 Code to the numerical example of a valuation portfolio

```

#zero coupon bond yield curves
R1 = c(0.0337, 0.0352, 0.0353, 0.0356, 0.036)
R2 = c(0.02, 0.0285, 0.029, 0.0296)
R3 = c(0.0069, 0.0184, 0.0214)
R4 = c(0.0058, 0.0079)
R5 = c(0.0099)

#calculates the put option price given the start time s, and yield curve
put = function(s,R){
    T = 2005
    t0 = 2000
    sigma = 0.15
    i = 0.03
    m = T-s
    j = s-t0+1
    I = c(1, 1.089806, 0.9626167, 0.8299416, 0.9726801)
    z = rep(NA,m)
    z[1:m] = exp(-c(1:m)*R[1:m])
    K = z*(1+i)^c(j:(T-t0))
    a1 = (-log(I[j]/K) + 0.5*(c(1:m))*sigma^2)/(sigma*sqrt(c(1:m)))
    a2 = a1 - sigma*sqrt(c(1:m))
    return(K*pnorm(a1) - I[j]*pnorm(a2))
}

#index
I = c(1, 1.089806, 0.9626167, 0.8299416, 0.9726801, 0.990644)
d = c(4,5,5,5,6)
l = c(1000, 996, 991, 986, 981, 975)

#calculates the premium
Pi = 100000*(d[1]*(1+put(2000,R1)[1]) + d[2]*(1+put(2000,R1)[2]) +
    d[3]*(1+put(2000,R1)[3])
    + d[4]*(1+put(2000,R1)[4]) + d[5]*(I[1]+put(2000,R1)[5]) + l[6])/
    (l[1]+ l[2]*exp(-1*R1[1]) + l[3]*exp(-2*R1[2])
    + l[4]*exp(-3*R1[3]) + l[5]*exp(-4*R1[4]))


PiR = rep(0,5)
Ep = rep(0,5)
Em = rep(0,5)

#calculates the market value of the valuation portfolio
Em[1] = -Pi*(l[2]*exp(-1*R1[1]) + l[3]*exp(-2*R1[2])
    + l[4]*exp(-3*R1[3]) + l[5]*exp(-4*R1[4])) + 100000*(d[1]*(I[1]+put(2000,R1)[1])
    + d[2]*(I[1]+put(2000,R1)[2]) + d[3]*(I[1]+put(2000,R1)[3])
    + d[4]*(I[1]+put(2000,R1)[4]) + d[5]*(I[1]+put(2000,R1)[5]) + l[6]*I[1])

```

```

Em[2] = -Pi*(l[3]*exp(-1*R2[1])
    + l[4]*exp(-2*R2[2]) + l[5]*exp(-3*R2[3])) + 100000*(
    d[2]*(I[2]+put(2001,R2)[1]) + d[3]*(I[2]+put(2001,R2)[2])
    + d[4]*(I[2]+put(2001,R2)[3]) + d[5]*(I[2]+put(2001,R2)[4]) + l[6]*I[2])


Em[3] = -Pi*(l[4]*exp(-1*R3[1]) + l[5]*exp(-2*R3[2])) + 100000*(
    d[3]*(I[3]+put(2002,R3)[1])
    + d[4]*(I[3]+put(2002,R3)[2]) + d[5]*(I[3]+put(2002,R3)[3]) + l[6]*I[3])


Em[4] = -Pi*(l[5]*exp(-1*R4[1])) + 100000*(
    d[4]*(I[4]+put(2003,R4)[1]) + d[5]*(I[4]+put(2003,R4)[2]) + l[6]*I[4])


Em[5] = 100000*(d[5]*(I[5]+put(2004,R5)[1]) + l[6]*I[5])

Ep[2:5] = Em[2:5] - l[2:5]*Pi

#calculates the reinsurance loading at time t0
reass = function(inc_mort){
PiR = rep(0,5)
loading = inc_mort - 1
PiR[1] = (loading*l[2]*(1-l[2]/l[1])) * (100000*(I[1] + put(2000,R1)[1]) + (Pi)*exp(-R1[1]) -
     (-Pi*(l[3]*exp(-2*R1[2])
    + l[4]*exp(-3*R1[3]) + l[5]*exp(-4*R1[4])) + 100000*(d[2]*(I[1]+put(2000,R1)[2]) +
        d[3]*(I[1]+put(2000,R1)[3])
    + d[4]*(I[1]+put(2000,R1)[4]) + d[5]*(I[1]+put(2000,R1)[5]) + l[6]*I[1]))/l[2])

PiR[2] = (loading*l[3]*(1-l[3]/l[2])) * (100000*(I[1] + put(2000,R1)[2]) + Pi*exp(-2*R1[2]) -
    (-Pi*(l[4]*exp(-3*R1[3]) + l[5]*exp(-4*R1[4])) +
        100000*(d[3]*(I[1]+put(2000,R1)[3])
    + d[4]*(I[1]+put(2000,R1)[4]) + d[5]*(I[1]+put(2000,R1)[5]) + l[6]*I[1]))/l[3])

PiR[3] = (loading*l[4]*(1-l[4]/l[3])) * (100000*(I[1] + put(2000,R1)[3]) + Pi*exp(-3*R1[3]) -
    (-Pi*(l[5]*exp(-4*R1[4])) +
        100000*(d[4]*(I[1]+put(2000,R1)[4]) + d[5]*(I[1]+put(2000,R1)[5]) + l[6]*I[1]))/l[4])

PiR[4] = (loading*l[5]*(1-l[5]/l[4])) * (100000*(I[1] + put(2000,R1)[4]) + Pi*exp(-4*R1[4]) -
    (100000*(d[5]*(1+put(2000,R1)[5])+l[6]))/l[4])

PiR[5] = loading*l[6]*(1-l[6]/l[5])*100000*(put(2000,R1)[5])
return(PiR)
}
```

D.2 Code to the simulation of the LIBOR forward rate

```
# Simulates 20 different paths
m = 1000

#simulates Brownian motion
b = function(m) {
    B = rep(0,m)
    B[1] = 0
    for(i in 2:m){
        B[i] = B[i-1] + rnorm(1,mean=0,sd=1)*sqrt(1/m)
    }
    return(B)
}

#Plots the simulations
go = function(sim){# sim = 20
    L1 = matrix(NA, nrow = (0.25*m), ncol = sim)
    L2 = matrix(NA, nrow = (0.5*m), ncol = sim)
    L3 = matrix(NA, nrow = (0.75*m), ncol = sim)
    for (i in 1:sim){
        B = b(m)
        L3[,i] = LIBOR3(B)
        L2[,i] = LIBOR2(B)
        L1[,i] = LIBOR1(B)
    }
    plots(L1, L2, L3, sim)
}

# Plotfunction
plots = function(L1, L2, L3, sim){
```

```
     #stepOne(L3, sim)
     #stepTwo(L2, sim)
     stepThree(L1, sim)
     #E()
}

#plots one simulations of each step up against each other.
sammen = function(){
     L1 = c(NA,(0.25*m))
     L2 = c(NA,(0.5*m))
     L3 = c(NA,(0.75*m))
     B = b(m)
     L3 = LIBOR3(B)
     L2 = LIBOR2(B)
     L1 = LIBOR1(B)
     pdf("sammen.pdf")
     plot(main = "The steps plotted against each other", lwd = 3, xlab = "Time steps", ylab = "LIBOR", L3, type = 'l', ylim = c(0.01,0.03))
     lines(L2, lwd = 2)
     lines(L1, lwd = 1)
     legend(0,0.03, legend = c("Step 1", "Step 2", "Step 3"), lty = c(1,1,1), lwd = c(3,2,1))
     dev.off()
}

#Plots step 1 sim times,
stepOne = function(L3, sim){
     pdf("Steg1.pdf")
     plot(main = "Step 1, 9 months ahead", xlab = "Time steps", ylab = "LIBOR",L3[,1], type = 'l', ylim = c(0.01,0.03))
     for (i in 2:sim){
          lines(L3[,i], type = 'l')
     }
     dev.off()
}

stepTwo = function(L2, sim){
     pdf("Steg2.pdf")
     plot(main = "Step 2, 6 months ahead", xlab = "Time steps", ylab = "LIBOR", L2[,1], type = 'l', ylim = c(0.01,0.03))
     for (i in 2:sim){
          lines(L2[,i], type = 'l')
     }
     dev.off()
}

stepThree = function(L1, sim){
     pdf("steg3.pdf")
     plot(main = "Step 3, 3 months ahead", xlab = "Time steps", ylab = "LIBOR", L1[,1], type = 'l', ylim = c(0.01,0.03))
     for (i in 2:sim){
          lines(L1[,i], type = 'l')
     }
     dev.off()
}

# m = 1000
# The same Brownian motion is used in all of the steps
# starting values found from bloomberg.com
LIBOR3 = function(B_T){
     L_03 = 1.85438/100
     L3 = rep(0,(0.75*m))
     for (i in 1:(0.75*m)) {
          L3[i] = L_03 * exp((16.05/100)*B_T[i] - (0.5 * (i/m) * (16.05/100)^2))
     }
     return(L3)
}

LIBOR2 = function(B_T) {
     L3 = LIBOR3(B_T)
     L_02 = 1.73688/100
     delta = 1/(4) #1/1000?
     L2 = rep(0,(0.5*m))
     int = 0
     for (i in 1:(0.5*m)) {
          int = int + ((16.12/100) * (-(delta*(16.05/100)*L3[i]) / (1+delta*L3[i])))
     }
     int = int / (0.5*m)
     for (i in 1:(0.5*m)) {
          L2[i] = L_02 * exp(((16.12/100) * B_T[i]) + int - (0.5 * (i/m) * (16.12/100)^2))
     }
     return(L2)
}
```

```
LIBOR1 = function(B_T){
    L2 = LIBOR2(B_T)
    L3 = LIBOR3(B_T)
    L1 = rep(0,(0.25*m))
    L_01 = 1.50400/100
    delta = 1/(4)
    int = 0
    for (i in 1:(0.25*m)){
        int = int + (16.19/100) * ((-delta * (16.12/100) * L2[i]) / (1 + delta*L2[i]) - (delta
            * (16.05/100) * L3[i]) / (1 + delta*L3[i]))
    }
    int = int/(0.25*m)
    for (i in 1:(0.25*m)){
        L1[i] = L_01 * exp(((16.19/100) * B_T[i]) + int - (0.5*(i/m)*(16.19/100)^2))
    }
    return(L1)
}
E = function(){
    L1 = 0
    L2 = 0
    L3 = 0
    for(i in 1:1000){
        L3 = L3 + LIBOR3(b(1000))
        L2 = L2 + LIBOR2(b(1000))
        L1 = L1 + LIBOR1(b(1000))
    }
    ###OBS! black = step 1, red = step 2, green = step 3
    pdf("forventning.pdf")
    plot(main = "Forventning_LIBOR", xlab = "Tidssteg", ylab = "LIBOR", L3/1000, type = 'l',
        ylim = c(0.015,0.025))
    lines(L2/1000, type = 'l', ylim = c(0.025,0.035), col = 'red')
    lines(L1/1000, type = 'l', ylim = c(0.025,0.035), col = 'green')
    dev.off()
    return(L3)
}

sd = function(){
    temp = mean(E())/1000
    sd3 = 0
        sd3 = sum((LIBOR3(b(1000) - temp)^2))
    return((1/1000-1)*sd3)
}
```

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